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Three-dimensional wave propagation in a general anisotropic poroelastic medium: phase velocity, group velocity and polarization

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SUMMARY

This is an attempt to study 3-D wave propagation in a general anisotropic poroelastic medium. Biot's theory is used to derive a modified Christoffel equation for the propagation of plane harmonic waves in an anisotropic fluid-saturated porous solid. This equation is solved further to obtain a biquadratic equation, the roots of which represent the phase velocities of all the four quasi-waves that may propagate in such a medium. These phase velocities vary with the direction of phase propagation. Expressions are derived to calculate the group velocities of all the four quasi-waves without using numerical differentiation. The eigensystem of modified Christoffel equation is used to calculate the polarizations of all the quasi-waves. The particle motion of each wave is a function of the direction of phase propagation. Some fundamental differences between wave propagation in anisotropic poroelastic medium and anisotropic elastic medium are suggested, an interesting one is that in an anisotropic poroelastic medium, the polarizations of different quasi-waves need not be mutually orthogonal. In the anisotropic poroelastic medium, the motion of fluid particles deviates from solid particles and this deviation varies, also, with the matrix porosity. Propagation regimes for an isotropic medium, giving velocities and polarizations of both compressional and shear waves, are obtained as special cases. The variations of phase velocity, group velocity, ray direction with phase direction (in 3-D space), are plotted for a numerical model of general anisotropic poroelastic solid. The same numerical model is used to plot the deviations of polarizations from phase direction and ray direction. The deviations among the motion in fluid of the particles and solid parts of porous aggregate are also plotted.

Key words: general anisotropy, group velocity, phase velocity, polarization, poroelasticity.

1 INTRODUCTION

Elastic wave propagation in anisotropic poroelastic media is of great interest in geophysics and other branches of applied sciences such as petroleum engineering, structural mechanics and seismology. In the absence of point symmetry of pores, a poroelastic solid behaves anisotropically to wave propagation. Stress-induced anisotropy may also be present in the matrix frame of a poroelastic solid. The seismic anisotropy in the earth is caused by the preferential alignments in crystal orientations, grains, pores, microcracks, regional fractures or joints. Such alignments are in direct response to the existing or former stress-field. Volumes of inherently isotropic rocks may be microfractured by regional tectonic stresses, resulting in a system of oriented fractures along principal axes which could lead to global anisotropy (Bamford & Nunn 1979; Leary *et al.* 1987). In exploration studies (Helbig 1984; Leary & Henyey 1985; Corrigan 1989; Kerner *et al.* 1989, and many others), velocity anisotropy measured from traveltimes has suggested the presence of significant anisotropy in sedimentary basins. Similar results for Earth's crust and mantle have been obtained in earthquake studies (Gupta 1973; Crampin 1985; Thomsen 1986; and many others). Crampin (1994) reviewed the various observational studies and confirmed the presence of shear wave splitting in almost all the rocks in the uppermost half of the crust.

The poroelastic equations formulated by Biot (1955, 1956) have long been regarded as standard and have formed the basis for solving wave propagation problems in poroelasticity. Burridge & Keller (1981) used homogenization theory (Levy 1979; Auriault 1980) to derive the equations which govern the linear macroscopic mechanical behaviour of a porous solid saturated with a compressible viscous fluid. However, for small viscosity, these equations are equivalent to those of Biot's formulation. Schmitt (1989) and Sharma (1991) used Biot's theory to study the wave propagation in transversely isotropic porous solids. Mengi & McNiven (1978) studied the propagation and decay of waves in isotropic and anisotropic poroelastic media, in the absence of fluid-solid coupling of Biot's theory. Theoretical models developed by Hudson (1980) are the bases of much of the work explaining the anisotropic wave propagation behaviour of cracked solids. These works, however, fail to account fully for the presence of both matrix porosity and crack porosity. Thomsen (1995) related the anisotropy to crack parameters in a porous rock and suggested that the amount and the type of anisotropy in the porous rocks depends upon the crack density, crack shape, stiffness of interstitial fluid, matrix porosity, frequency, fluid pressure and flow between cracks and pores. Rathore *et al.* (1995) supported this work through the experimental study of anisotropy in a sandstone with controlled crack geometry. Sharma (1996) discussed the co-existence of cracks and pores and its effect on surface wave propagation. Hudson *et al.* (1996) studied the effect of interconnection between cracks and of small-scale porosity within the solid material on the overall elastic properties of cracked solid. Very recently, Tod (2002) discussed the changes in the crack distribution and fluid flow properties of the aggregate with the alteration in applied stress and fluid pressure. The expressions for effective elastic constants, involving a dependence on both the applied stress and the fluid pressure, are derived and are used to determine their effects on the anisotropy of the effective medium.

An important tool for studying the properties of anisotropic elastic wave propagation is the capability to model slowness and polarization. Of these, the variations of velocities with direction are often considered to be the dominant indicator of anisotropy. However, the velocity variations that are averaged over very long paths are not of much use in studying a local structure. The polarizations of direct arrivals are more sensitive to the local properties of a medium. So, within a few wavelengths of the recording station, the diagnostic effect of anisotropy is better represented by the polarization anomalies than velocity dispersion. Moreover, the complicated structure of the Earth inhibits our ability to obtain accurate measurements of velocity over a range of directions in one plane of a material. This makes velocity anisotropy difficult to observe. An easily observable and distinctive feature of wave propagation in anisotropic solids is the deviation of particle displacements from the ray path. These anomalies contain information concerning the nature of anisotropic alignments and in some cases may indicate the depth of the anisotropic layer. Variations of delays in anisotropic polarization anomalies are used to extract information concerning the orientations of stress-induced dilatancy and also concerning the state of stress causing dilatancy (Crampin & McGonigle 1981). Polarization vectors of waves propagating in anisotropic media are used to calculate Green's tensor which plays an important role in the computation of displacements from point sources (Aki & Richards 1980; Ben-Menahem & Singh 1981). Ben-Menahem *et al.* (1991) have shown that the radiation patterns are controlled entirely by the dot product of polarization dyadic and point source vector. Sayers (1988) used the inversion of ultrasonic velocity and polarizations to obtain crack orientation distribution function and verified the results with those obtained by petrofabric analysis. Meadows & Winterstein (1994) studied the use of polarizations of *S* waves in detecting the hydraulic fracture at Lost Hills Field, California.

In the available literature, all the analytical studies on anisotropic propagation in poroelastic solids restrict the motion to a fixed (symmetry or arbitrary) plane and hence solve a 2-D problem. The energy propagation in an anisotropic media is, in fact, a 3-D phenomenon. Moreover, the mineral orientations, microfracturing or thin layering or combinations of these can disturb the point symmetry of pores that results in an anisotropy of arbitrary symmetry. The work presented studies both velocities and polarizations for the 3-D wave propagation in a general anisotropic poroelastic medium. The analytical expressions are derived for phase velocity, group velocity, ray direction and polarizations of all the four quasi-waves.

2 ANISOTROPIC POROELASTICITY

Following Biot (1956), the governing equations for a fluid-saturated porous media, in the absence of body forces and dissipation, are

$$\begin{aligned} \sigma_{ij,j} &= \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i; \\ \sigma_{,i} &= \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i. \end{aligned} \tag{1}$$

In these equations, u_i and U_i are the components of the average displacements for the solid and fluid phases, respectively. The dot notation is used to represent differentiation with respect to time. Indices can take the values 1, 2 and 3. Summation convention is valid for repeated indices. The comma before an index represents partial space differentiation. ρ_{11}, ρ_{12} and ρ_{22} are the dynamic constants related to the porosity of solid (f), densities of solid particles and interstitial fluid (ρ_s, ρ_f), through the relations: $\rho_{11} + \rho_{12} = (1 - f)\rho_s$, $\rho_{12} + \rho_{22} = f\rho_f$. For an anisotropic porous material, the constitutive equations for stresses in the solid phase (i.e. σ_{ij}) and fluid (i.e. σ) are

$$\begin{aligned} \sigma_{ij} &= c_{\mathrm{ijkl}}u_{k,l} + m_{ij}U_{k,k}; \\ \sigma &= m_{ij}u_{i,j} + RU_{k,k}. \end{aligned} \tag{2}$$

The coefficients c_{ijkl}, m_{ij} and R are the material constants of a linear porous material. The assumptions $c_{\mathrm{ijkl}} = c_{\mathrm{klij}} = c_{\mathrm{jikl}}$ and $m_{ij} = m_{ji}$ reduce the number of independent material constants in the constitutive equations for a general anisotropy to, at the most, 28.

To seek the harmonic solution of (1), for the propagation of plane waves, write

$$\begin{aligned} u_j &= S_j \exp\left[i\omega(p_k x_k - t)\right]; \\ U_j &= F_j \exp\left[i\omega(p_k x_k - t)\right], \quad (j = 1, 2, 3), \end{aligned} \tag{3}$$

where, ω is frequency and (p_1, p_2, p_3) is slowness vector. Substituting (2) in (1) and then using (3), we obtain a system of six homogeneous equations in S_1, S_2, S_3, F_1, F_2, F_3. Non-trivial solution of this system of equations defines the modified Christoffel equation for the wave propagation in an anisotropic poroelastic medium. Eliminating F_j, ($j = 1, 2, 3$), the modified Christoffel equation is a system of three homogeneous equations, given by

$$W_{ij}S_j = 0, \quad (i = 1, 2, 3). \tag{4}$$

The elements W_{ij} of square matrix of order 3 are defined as

$$W_{ij} = -g_0 h \delta_{ij} + P_{ij} + \frac{1}{h-1} Q_{ij}, \quad (i, j = 1, 2, 3), \tag{5}$$

where, δ_{ij} is Kronecker delta. $\mathbf{P}$ and $\mathbf{Q}$, the square matrices of order 3, are defined as follows.

The row matrix $\mathbf{N} = (n_1, n_2, n_3)$, where n_j denotes the components of a unit vector normal to wave surface, represents the direction of phase propagation. In terms of phase velocity v, the slowness is, then, $\{p_1, p_2, p_3\} = \{n_1, n_2, n_3\}/v$. Consider a general anisotropic poroelastic medium with elastic constants c_{ijkl} of the solid matrix represented by two-suffix notation, c_{ij}. Define, following Sharma (2002),

$$\begin{aligned} \alpha &= \mathbf{NAN}', \quad \beta = \mathbf{NBN}', \quad \gamma = \mathbf{NCN}', \\ \delta &= \mathbf{NDN}', \quad \eta = \mathbf{NEN}', \quad \zeta = \mathbf{NFN}', \end{aligned} \tag{6}$$

where $\mathbf{N}'$ denotes the transpose of row matrix $\mathbf{N}$. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{F}$ are square matrices of order 3. For general anisotropy, these are defined as follows.

$$\begin{aligned} \mathbf{A} &= \{a_{11}, a_{16}, a_{15}; a_{16}, a_{66}, a_{56}; a_{15}, a_{56}, a_{55}\}; \\ \mathbf{B} &= \{a_{66}, a_{26}, a_{46}; a_{26}, a_{22}, a_{24}; a_{46}, a_{24}, a_{44}\}; \\ \mathbf{C} &= \{a_{55}, a_{45}, a_{35}; a_{45}, a_{44}, a_{34}; a_{35}, a_{34}, a_{33}\}; \\ \mathbf{D} &= \{a_{16}, a_{12}, a_{14}; a_{66}, a_{26}, a_{46}; a_{56}, a_{25}, a_{45}\}; \\ \mathbf{E} &= \{a_{15}, a_{14}, a_{13}; a_{56}, a_{46}, a_{36}; a_{55}, a_{45}, a_{35}\}; \\ \mathbf{F} &= \{a_{56}, a_{46}, a_{36}; a_{25}, a_{24}, a_{23}; a_{45}, a_{44}, a_{34}\}, \end{aligned} \tag{7}$$

where $a_{ij} = c_{ij}/R$. The matrix $\mathbf{P} = \mathbf{Z} + \mathbf{Y}$ and matrix $\mathbf{Q} = \{Q_{ij}\}$ is defined by $Q_{ij} = X_i X_j + Y_{ij}$, where symmetric, square matrix

$$\mathbf{Z} = \{\alpha, \delta, \eta; \delta, \beta, \zeta; \eta, \zeta, \gamma\}, \tag{8}$$

and the elements of symmetric matrix $\mathbf{Y}$ are

$$Y_{ij} = r_{12}^2 n_i n_j - r_{12}(n_i X_j + n_j X_i), \quad (i, j = 1, 2, 3). \tag{9}$$

The elements of row matrix $\mathbf{X}$ are defined as $X_i = m_{ik} n_k / R$, $(i = 1, 2, 3)$. In eq. (5), the variable $h = \rho_{22} v^2 / R$ and $g_0 = r_{11} - r_{12}^2$, where $r_{1j} = \rho_{1j}/\rho_{22}$, $(j = 1, 2)$. The system of eqs (4) is possible for all values of h other than 0 and 1. The non-zero value of h implies that this system of equations for saturated poroelastic solid cannot be reduced to the one for dry poroelastic solid. $h = 1$ represents the case when phase velocity assumes the value of $\sqrt{R/\rho_{22}}$.

3 PHASE VELOCITY

For the non-trivial solution of the system of eqs (4), the determinant of matrix $\mathbf{W}$ must vanish. This gives a biquadratic equation in h $(=\rho_{22} v^2/R)$, given by

$$h^4 - c_1 h^3 + c_2 h^2 - c_3 h + c_4 = 0. \tag{10}$$

Unlike in an anisotropic elastic medium, this biquadratic equation is not derived from the eigensystem of a real, symmetric positive definite matrix. This implies that roots of this equation may not, always, be positive. Therefore, the number of waves propagating in such a medium depends upon the coefficients of this biquadratic equation. For all the four body waves to propagate in an anisotropic poroelastic medium, all the coefficients (i.e. c_j, $j = 1, 2, 3, 4$) must be positive. Let h_j, $(j = 1, 2, 3, 4)$, denote the four (positive) roots of this equation. The velocities of the four waves, given by $v_j = \sqrt{(R h_j/\rho_{22})}$, $(j = 1, 2, 3, 4)$, will be varying with the direction of phase propagation and called phase velocities. These waves are called quasi-waves because polarizations may not be along the dynamic axes. Keeping in mind the propagation of two compressional and two shear waves in an isotropic poroelastic medium (Appendix A), these waves represented by $j = 1, 2, 3$ and 4, are called the qP1, qP2, qS1 and qS2 waves, respectively. The coefficients of eq. (10) are as follows (a repeated index implies summation).

$$\begin{aligned} c_1 &= P_{ii}/g_0 + 1; \\ c_2 &= (P_{ii} - Q_{ii})/g_0 - T_1/g_0^2; \\ c_3 &= (\det\{P_{ij}\}/g_0 + T_2 - T_1)/g_0^2; \\ c_4 &= (\det\{P_{ij}\} - T_3)/g_0^3, \end{aligned} \tag{11}$$

where

$$\begin{aligned} T_1 &= P_{12}^2 + P_{13}^2 + P_{23}^2 - P_{11}P_{22} - P_{11}P_{33} - P_{22}P_{33}, \\ T_2 &= 2(P_{12}Q_{12} + P_{13}Q_{13} + P_{23}Q_{23}) - Q_{11}(P_{22} + P_{33}) - Q_{22}(P_{11} + P_{33}) - Q_{33}(P_{11} + P_{22}), \\ T_3 &= Q_{11}\left(P_{22}P_{33} - P_{23}^2\right) + Q_{22}\left(P_{11}P_{33} - P_{13}^2\right) + Q_{33}\left(P_{11}P_{22} - P_{12}^2\right) \\ &\quad + 2Q_{12}\left(P_{13}P_{23} - P_{12}P_{33}\right) + 2Q_{13}\left(P_{12}P_{23} - P_{13}P_{22}\right) + 2Q_{23}\left(P_{12}P_{13} - P_{11}P_{23}\right). \end{aligned} \tag{12}$$

The roots of eq. (10) are written as

$$h_1 = 0.5(-G - L + \Delta_1);$$
$$h_2 = 0.5(-G - L - \Delta_1);$$
$$h_3 = 0.5(-G + L + \Delta_2);$$
$$h_4 = 0.5(-G + L - \Delta_2), \quad (13)$$

where

$$\Delta_1 = \sqrt{(G+L)^2 - 4(H+M)}; \qquad \Delta_2 = \sqrt{(G-L)^2 - 4(H-M)};$$
$$G = -0.5c_1; \qquad H = \sqrt{\left(c_2^2/9 - c_1c_3/3 + 4c_4/3\right)}\cos\psi + c_2/6;$$
$$M = \sqrt{H^2 - c_4}; \qquad L = (0.5c_3 + GH)/M; \qquad (L = \sqrt{G^2 - c_2 + 2H}; \quad \text{if } M = 0);$$
$$\psi = \frac{1}{3}\tan^{-1}(\Delta/\Gamma); \qquad \Delta = \sqrt{-\Gamma^2 + \left(c_2^2/9 - c_1c_3/3 + 4c_4/3\right)^3};$$
$$\Gamma = 0.5\left[c_3^2 + 2c_2^3/27 - c_2(c_1c_3 - 4c_4)/3 - c_4\left(4c_2 - c_1^2\right)\right].$$

4 GROUP VELOCITY

In an anisotropic medium, energy travels with group velocity along a ray at an angle to the propagation direction. It is the group velocity that is measured in observations of arrival times. The group velocity is also required for the interpretation of both real and synthetic data. In a spherical coordinate system (r, θ, ϕ), let $v_j(\theta, \phi)$ define the phase velocity of wave j in the direction of $\mathbf{N}$ $(=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in the Cartesian coordinate system). Ray direction is determined from the components of group velocity. These components w_j, $(j = 1, 2, 3)$,

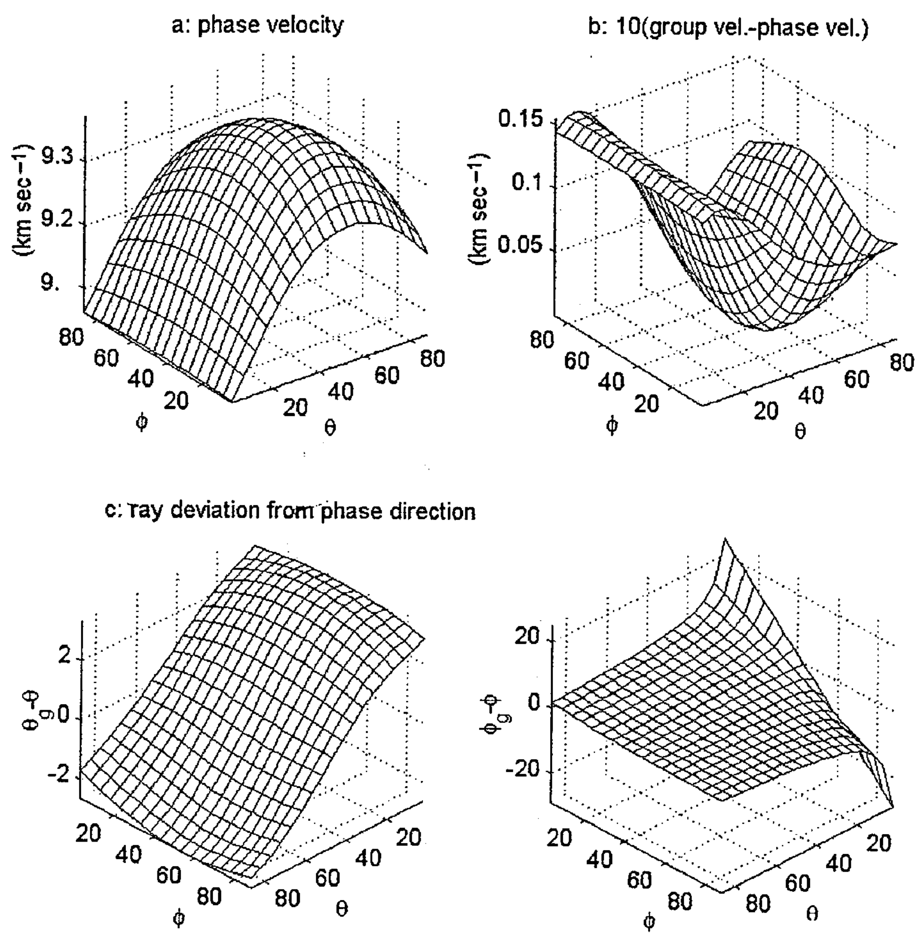

Figure 1. Variations of phase velocity, group velocity and ray direction of the qP1 wave with phase direction (θ, ϕ); all angles are in degrees.

following Ben-Menahem & Sena (1990), are expressed as follows.

$$w_x/v_j = \cos\phi\sin\theta + \cos\phi\cos\theta T_\theta - \frac{\sin\phi}{\sin\theta}T_\phi;$$
$$w_y/v_j = \sin\phi\sin\theta + \sin\phi\cos\theta T_\theta + \frac{\cos\phi}{\sin\theta}T_\phi;$$
$$w_z/v_j = \cos\theta - \sin\theta T_\theta. \tag{14}$$

The magnitude of the group velocity is

$$w_j = v_j\sqrt{1 + T_\theta^2 + \frac{1}{\sin^2\theta}T_\phi^2}; \tag{15}$$

and the ray direction, (θ_g, ϕ_g), is given by

$$\theta_g = \tan^{-1}\left(\frac{\sqrt{w_x^2 + w_y^2}}{w_z}\right); \qquad \phi_g = \tan^{-1}\left(\frac{w_y}{w_x}\right). \tag{16}$$

T_θ and T_ϕ in (14)–(15), for each wave $j = 1, 2, 3, 4$, are defined by,

$$T_k = \frac{1}{v_j}(v_j)_{,k} = \frac{1}{2h_j}(h_j)_{,k}; \quad (k = \theta, \phi). \tag{17}$$

The partial derivatives of h_j, $(j = 1, 2, 3, 4)$, for each of the four quasi-waves, derived from eq. (10), are given by the relation

$$h' = \frac{c_1'h^3 - c_2'h^2 + c_3'h - c_4'}{4h^3 - 3c_1h^2 + 2c_2h - c_3}. \tag{18}$$

The derivatives of coefficients (i.e. c'_k, $k = 1, 2, 3$) can be derived analytically from the relations defined in Section 3.

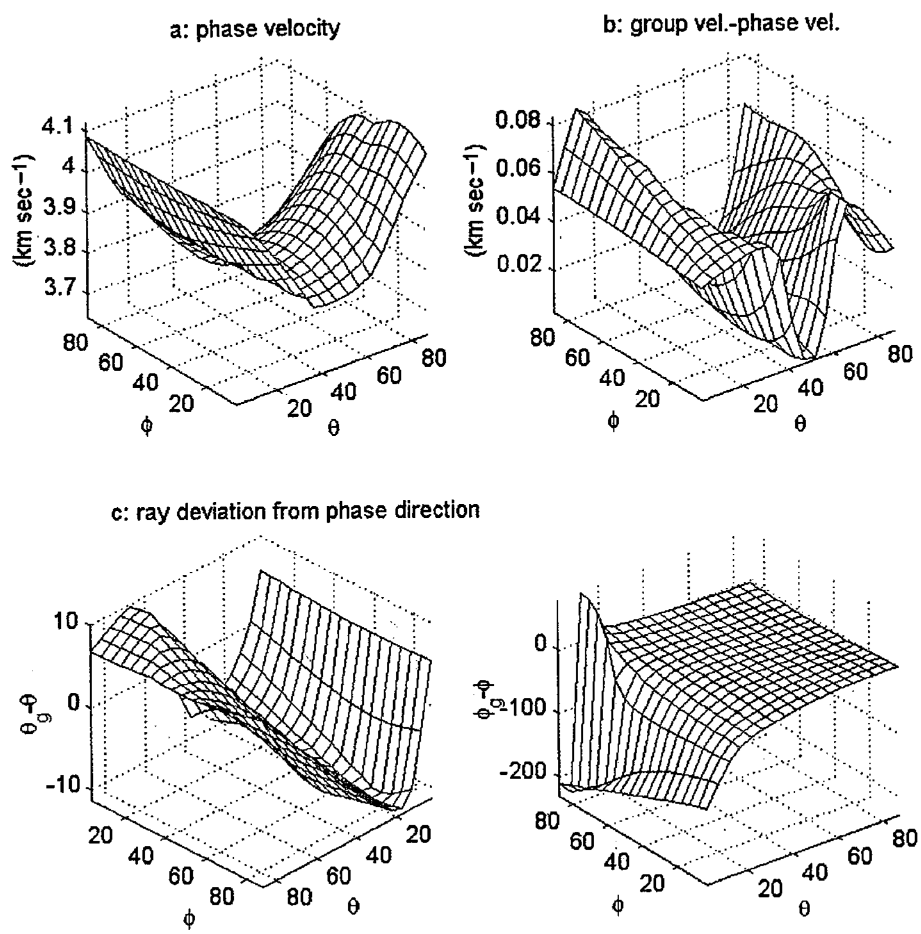

Figure 2. Variations of phase velocity, group velocity and ray direction of the qP2 wave with phase direction (θ, ϕ); all angles are in degrees.

5 POLARIZATION

The modified Christoffel eq. (4) is written as a system of three homogeneous equations, given by

$$W_{ij}u_j = 0, \quad (i = 1, 2, 3). \tag{19}$$

The fluid displacement U_i is related to solid displacement u_i by the relation

$$U_i = G_{ij}u_j, \tag{20}$$

where

$$G_{ij} = -r_{12}\delta_{ij} + \frac{1}{h-1}\left(\frac{1}{R}m_{ij} - r_{12}\delta_{ij}\right)n_i n_j. \tag{21}$$

The system of eqs (19) and that the components u_1, u_2, u_3 must satisfy the relations

$$\frac{u_1}{\Gamma_1} = \frac{u_2}{\Gamma_2} = \frac{u_3}{\Gamma_3}, \tag{22}$$

for three sets of $(\Gamma_1, \Gamma_2, \Gamma_3)$. These sets of values of Γ_i, $(i = 1, 2, 3)$, are as follows.

$$\text{(i) } \Gamma_1 = \Omega_2 \qquad \Gamma_2 = \Omega_1 \qquad \Gamma_3 = \Omega_{12}, \tag{23}$$

$$\text{(ii) } \Gamma_1 = \Omega_3 \qquad \Gamma_2 = \Omega_{13} \qquad \Gamma_3 = \Omega_1; \tag{24}$$

$$\text{(iii) } \Gamma_1 = \Omega_{23} \qquad \Gamma_2 = \Omega_3 \qquad \Gamma_3 = \Omega_2, \tag{25}$$

where

$$\begin{aligned}\Omega_1 &= P_{23}g_0h^2 + [P_{12}P_{13} - P_{11}P_{23} + g_0(Q_{23} - P_{23})]h \\ &\quad + P_{12}Q_{13} - P_{11}Q_{23} + P_{13}Q_{12} - P_{23}Q_{11} + P_{11}P_{23} - P_{12}P_{13}, \\ \Omega_{23} &= g_0^2h^3 - g_0(P_{22} + P_{33} + g_0)h^2 + \left[P_{22}P_{33} - P_{23}^2 + g_0(P_{22} + P_{33} - Q_{22} - Q_{33})\right]h \\ &\quad + P_{22}Q_{33} + P_{33}Q_{22} - 2P_{23}Q_{23} + P_{23}^2 - P_{22}P_{33}.\end{aligned} \tag{26}$$

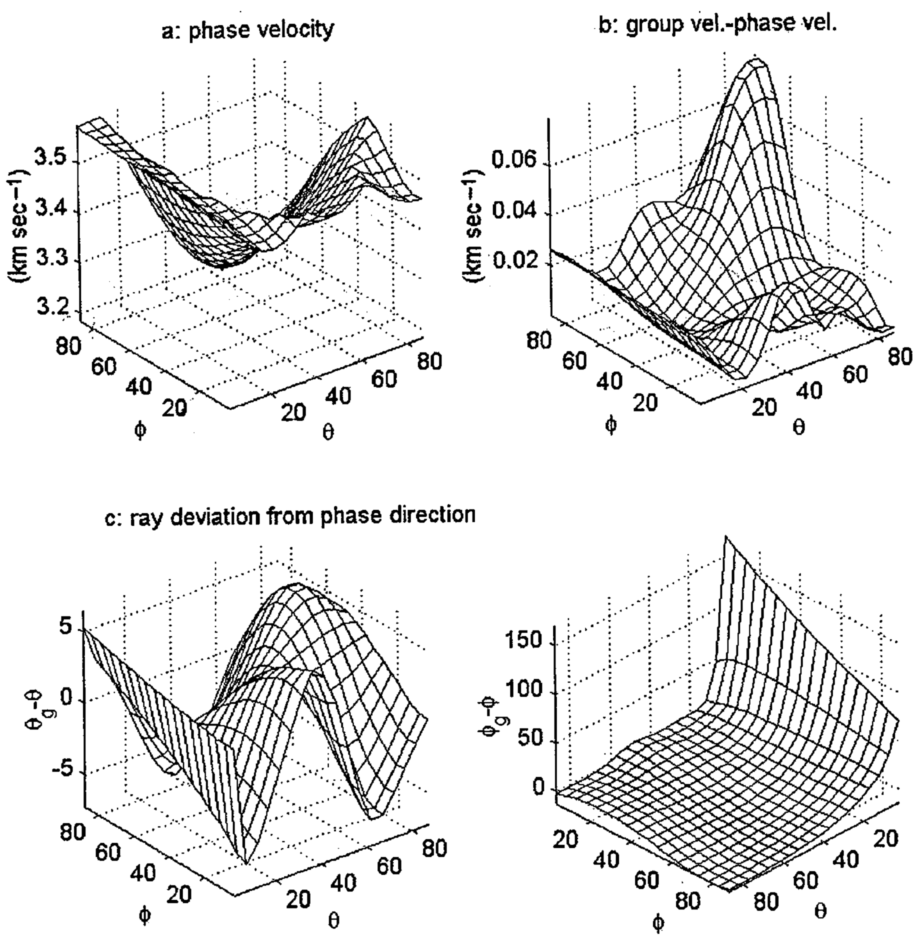

Figure 3. Variations of phase velocity, group velocity and ray direction of the qS1 wave with phase direction (θ, ϕ); all angles are in degrees.

The expression for $\Omega_2(\Omega_3)$ can be obtained from that of Ω_1 by interchanging the indices 1 and 2 (3). Similarly, the expression for $\Omega_{12}(\Omega_{13})$ can be obtained from that of Ω_{23} by replacing the index 1 with 3 (2).

Polarizations can be obtained from any of the three sets of Γ_i. The extra degree of freedom is removed by normalization. These sets are not independent but all three are required to find the polarizations for u_i, $(i = 1, 2, 3)$ in two situations. These situations (Fryer & Frazer 1987) are indeterminate solutions and singularities. Indeterminate solutions occur when along a particular phase direction, all Γ_i, $(i = 1, 2, 3)$, from one of the relations among (23) to (25) vanish. Then, u_i calculated from the zero Γ_i will be a null vector. However, fortunately, one of the remaining two sets of Γ_i will always provide a non-trivial solution for polarizations of u_i. Polarizations of U_i are, then, calculated from the relation (20). Singularities are the phase directions along which more than one quasi-waves have a common phase velocity. In such a situation, a single set of Γ_i will assign the same polarizations for these waves and hence represent the same wave. So, in the vicinity of a singularity, other sets of Γ_i help to find the polarizations of the unrepresented waves. Singularities are unlikely for the qP1 wave because of its large phase velocity as compared with other quasi-waves. To find the polarization in the case of singularities, the algorithm given by Fryer & Frazer (1987) may be used. A detailed study of singularities for anisotropic wave propagation in a poroelastic medium (analogous to singularities in anisotropic elastic medium) may be the subject of further studies.

There are some fundamental differences between wave propagation in an anisotropic elastic medium and an anisotropic poroelastic medium. These are as follows.

(1) Phase velocities of waves propagating in general anisotropic elastic medium are the eigenvalues of a real symmetric positive definite matrix and this ensures the propagation of exactly three waves along all phase directions (Sharma 2002). In an anisotropic poroelastic medium, the propagation of all the four quasi-waves along all phase directions is not guaranteed.

(2) Polarization vectors in an anisotropic elastic medium are the eigenvectors of a real symmetric positive definite matrix and hence will be mutually orthogonal. In an anisotropic poroelastic medium, the polarization vectors of different quasi-waves may not be orthogonal.

(3) In an anisotropic poroelastic medium, the deviation of fluid particles from solid particles varies with the phase direction. The polarizations of fluid particles have no role to play in the wave propagation in anisotropic elastic solids even when the anisotropy is caused by the fluid-filled cracks.

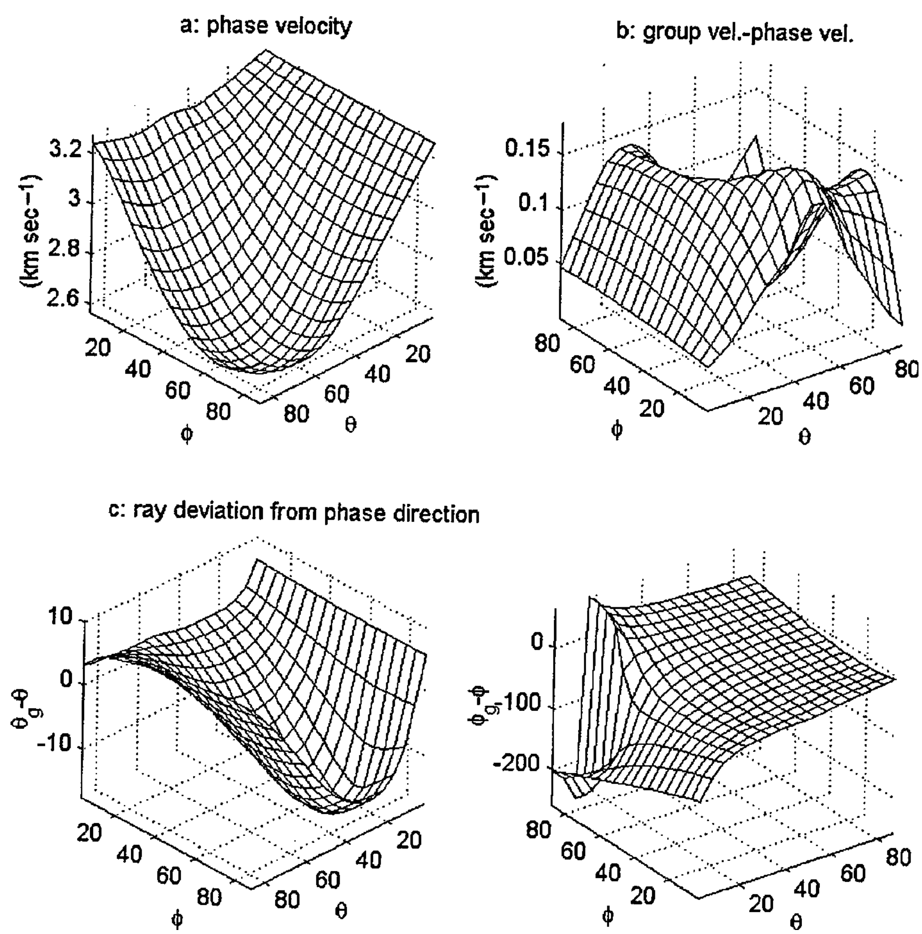

Figure 4. Variations of phase velocity, group velocity and ray direction of the qS2 wave with phase direction (θ, ϕ); all angles are in degrees.

6 PARTICLE MOTION

Polarization relations derived in Section 5 give the direction of motion of both solid and fluid particles of the porous aggregate. For each of the quasi-waves, (i.e. for $h = h_j$, $(j = 1, 2, 3, 4)$), (S_1, S_2, S_3), the unnormalized polarization vector for the displacement of solid particles, is obtained from the relations

$$\frac{S_1}{\Gamma_1} = \frac{S_2}{\Gamma_2} = \frac{S_3}{\Gamma_3}. \tag{27}$$

The direction (θ_s, ϕ_s) of motion of solid particles is given by

$$\theta_s = \tan^{-1}\left(\frac{\sqrt{\Gamma_1^2 + \Gamma_2^2}}{\Gamma_3}\right), \qquad \phi_s = \tan^{-1}\left(\frac{\Gamma_2}{\Gamma_1}\right). \tag{28}$$

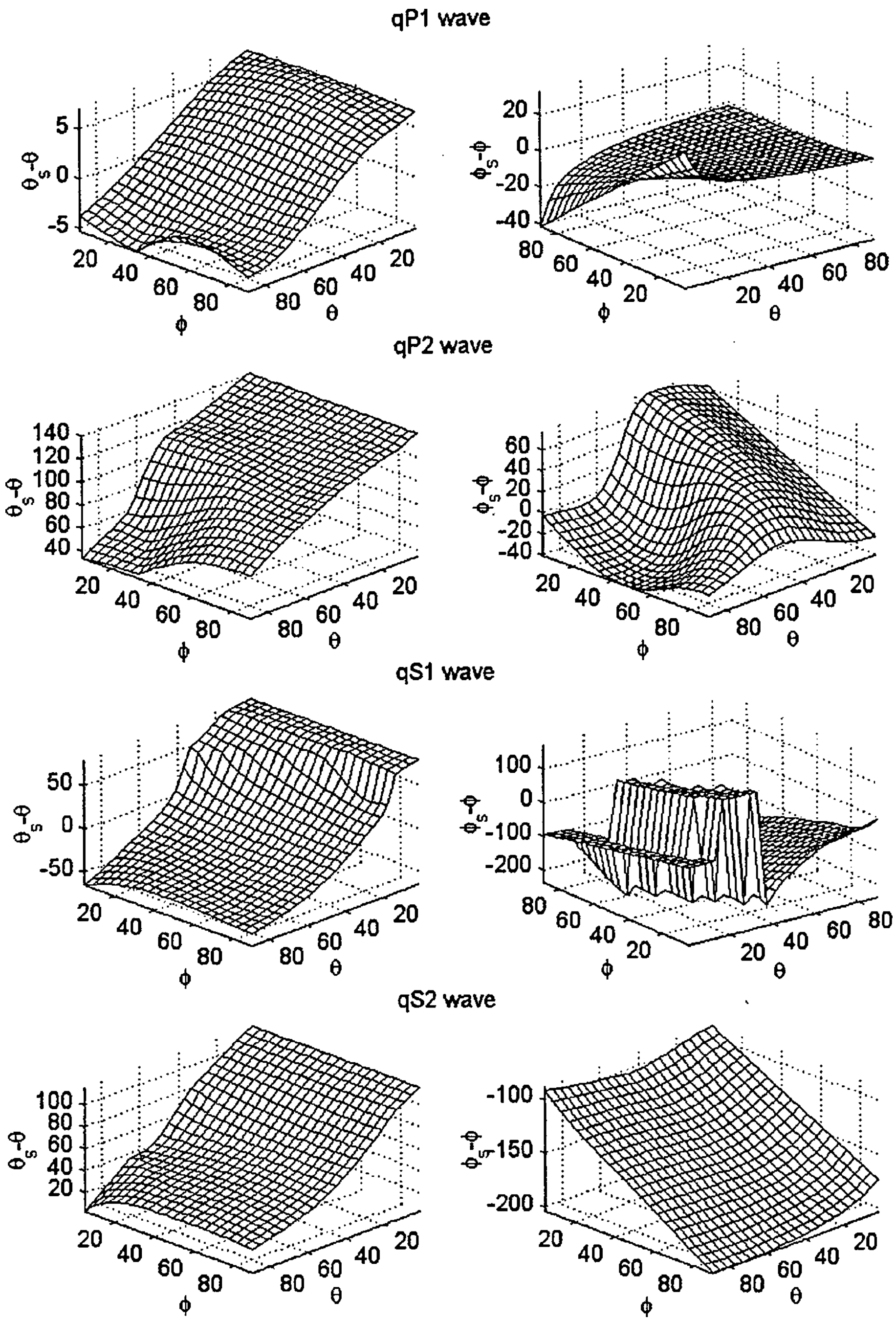

Figure 5. Variations of deviation (from phase direction) of solid particle polarization with phase direction (θ, ϕ); all angles in degrees.

Similarly, the direction (θ_f, ϕ_f) of displacement of fluid particles is given by

$$\theta_f = \tan^{-1}\left(\frac{\sqrt{\Delta_1^2+\Delta_2^2}}{\Delta_3}\right), \qquad \phi_f = \tan^{-1}\left(\frac{\Delta_2}{\Delta_1}\right), \tag{29}$$

where

$$\Delta_i = G_{ij}\Gamma_j, \quad (i = 1, 2, 3). \tag{30}$$

7 NUMERICAL COMPUTATION AND DISCUSSION

The analytical expressions derived in the previous sections represent a mathematical model for anisotropic propagation in a poroelastic medium. In a sense, it is a restricted study that considers the anisotropic wave propagation but without going into the cause of anisotropy. The numerical computation is, therefore, restricted to discuss the directional variations of velocities and polarizations. In a general anisotropic medium, the 3-D propagation depends upon azimuth also. Therefore, variations in phase direction is represented by both, the polar angle and

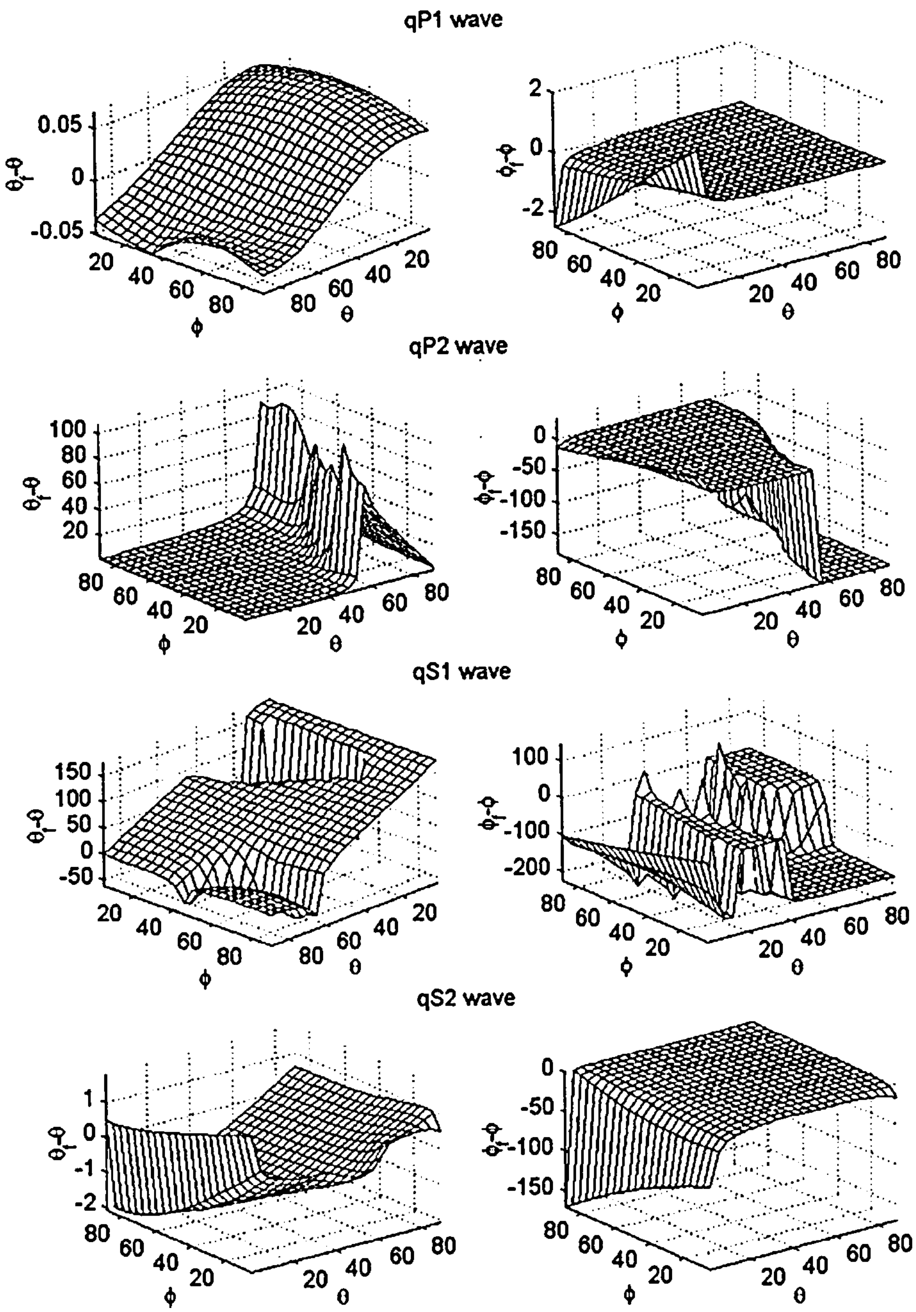

Figure 6. Variations of deviation (from phase direction) of fluid particles with phase direction (θ, ϕ); all angles in degrees.

the azimuth. The variations in these angles are considered from 0° to 90°. The analysis of anisotropic propagation in a real crustal material may be a useful study. Dolomite, a reservoir rock is chosen as the numerical model of an anisotropic poroelastic medium. Following, Rasolofosaon & Zinszner (2002), the elastic matrix (GPa) for dolomite, is given by

$c_{11} = 65.53 c_{12} = 9.77 c_{13} = 12.19 c_{14} = 0.18 c_{15} = -0.81 c_{16} = 2.94;$

$c_{22} = 50.77 c_{23} = 11.61 c_{24} = -0.09 c_{25} = -0.50 c_{26} = -0.19;$

$c_{33} = 60.11 c_{34} = -1.61 c_{35} = 1.78 c_{36} = 0.84;$

$c_{44} = 23.51 c_{45} = 1.49 c_{46} = -1.17;$

$c_{55} = 24.57 c_{56} = 0.26 c_{66} = 20.21$

The density is 2423 kg m^{-3}. The values assumed for remaining elastic parameters (GPa) are $\{m_{11}, m_{22}, m_{33}, m_{12}, m_{13}, m_{23}\} = \{20, 21, 19, 1, 2, 2.5\}$ and $R = 15$. Dynamic constants are derived for 23 per cent porosity, in a solid of density 2.423 g cc^{-1} and containing a fluid of

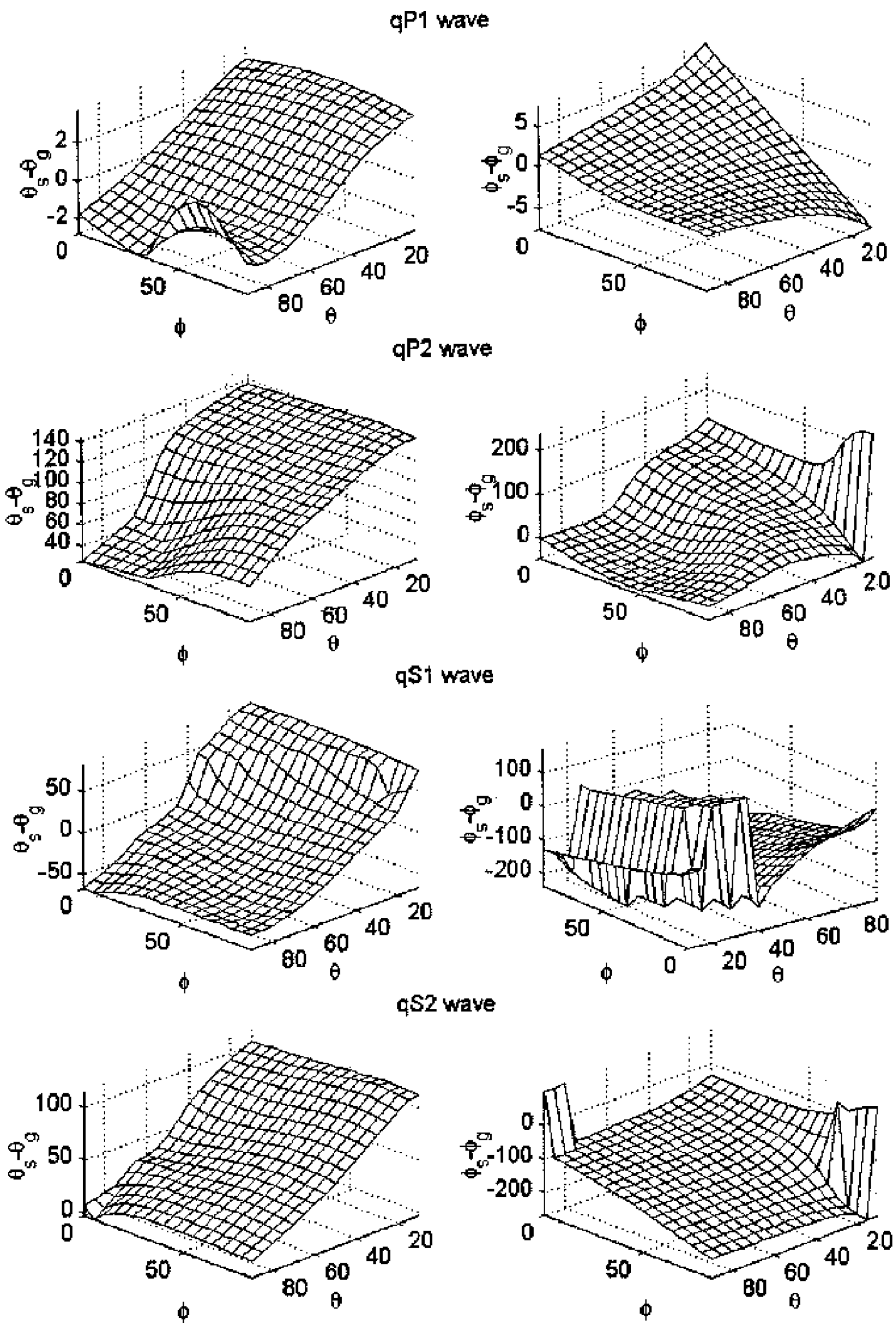

Figure 7. Variations of apparent polarizations of solid particles with phase direction (θ, ϕ); all angles in degrees.

density 0.98 g cc^{-1}. According to Biot's theory (1956), ρ_{12} represents fluid-solid coupling and its value is, generally, very small and negative. For an assumed value of $\rho_{12} = -0.01$, $\rho_{11} = (1 - 0.23)2.423 - \rho_{12}$ and $\rho_{22} = 0.23(0.98) - \rho_{12}$.

7.1 Velocities

Using the above numerical values, the variations of velocities with phase direction are presented in Figs 1–4 , for the qP1, qP2, qS1 and qS2 waves, respectively. Each of the Figs 1–4 exhibits the variations of (1) phase velocity, (2) difference between group velocity and phase velocity and (3) deviation of ray direction (polar angle and azimuth) from phase direction, with the phase direction. For the qP1 wave, as shown in Fig. 1, the phase velocity is minimum when $\theta = 0$. The effect of azimuth variations on phase velocity is negligible for smaller values of θ, but important at the larger values. The group velocity of the qP1 wave is very close to its phase velocity. The difference between group velocity and phase velocity is relatively larger when propagation is nearly along the polar axis (i.e. $\theta \approx 0$). The deviation of ray direction from

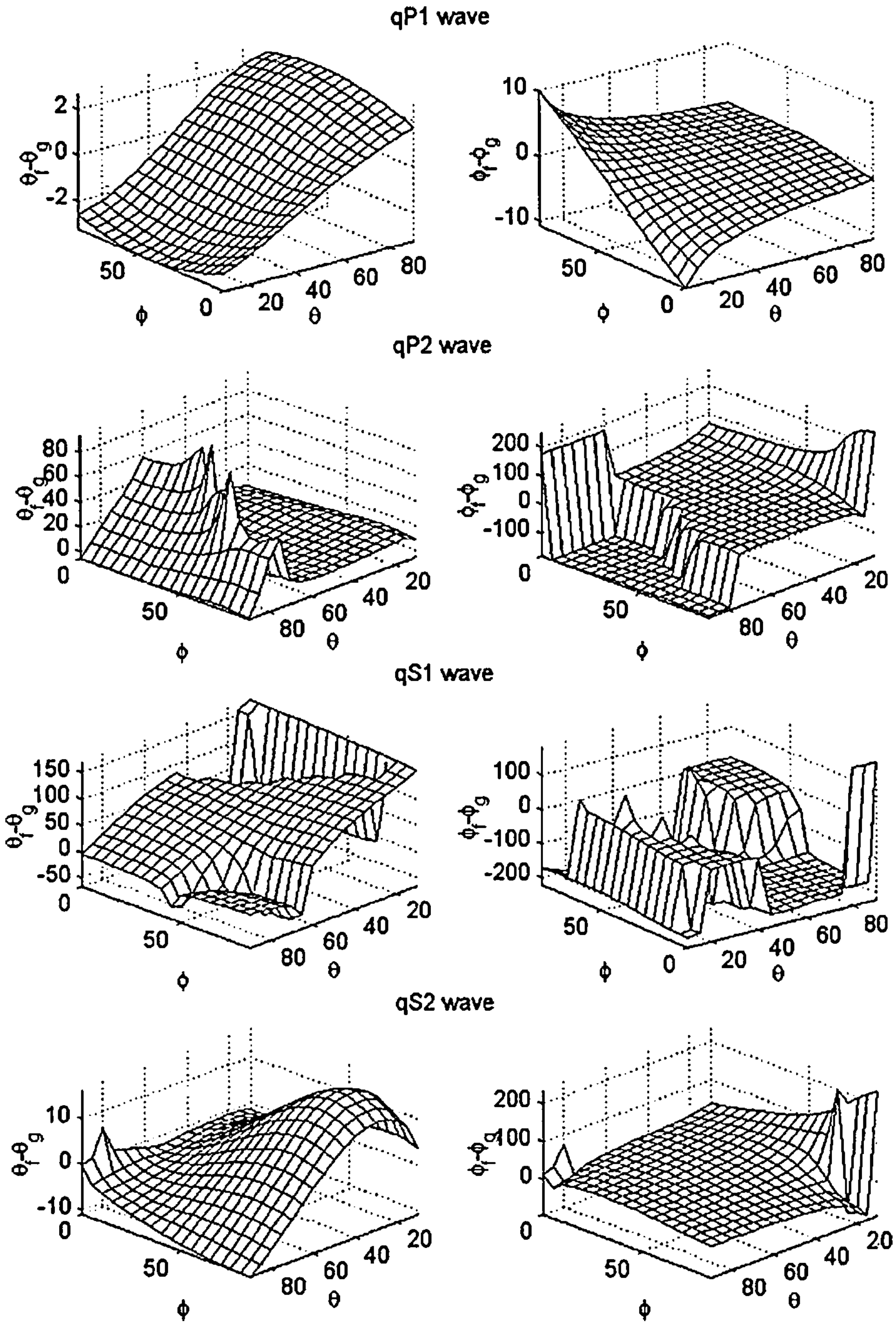

Figure 8. Variations of apparent polarizations of fluid particles with phase direction (θ, ϕ); all angles in degrees.

phase direction is larger in azimuth as compared with polar angle. The deviation in azimuth is maximum at $(\theta, \phi) = (0, 0)$ and minimum at $(0°, 90°)$. The velocity variations for the qP2 wave are plotted in Fig. 2. The group velocity of this wave has larger variations (≈2 per cent of phase velocity) as compared with qP1 wave (≈0.2 per cent). The deviation range of ray direction from phase direction is much smaller for polar angle than azimuth. Figs 3 and 4 contain the variations for the qS1 and qS2 waves, respectively. The difference between group velocity and phase velocity is smaller (≈2 per cent of phase velocity) for the qS1 wave as compared with qS2 waves (≈5 per cent). Deviation of polar angle of ray from θ is around 5°–10°, whereas azimuth of ray deviates up to 180°. For phase propagation away from the polar axis, the velocities vary significantly with the azimuth of the phase. The deviation of ray direction from phase direction also varies with the azimuth of phase propagation. The variations of velocities with the azimuth, in all plots, exhibit the importance of azimuthal anisotropy for the wave propagation in anisotropic poroelastic solids.

As a special case, the phase velocities of quasi-waves were calculated numerically for the 2-D wave propagation in transversely isotropic poroelastic solid and were verified with those obtained in Sharma (1991).

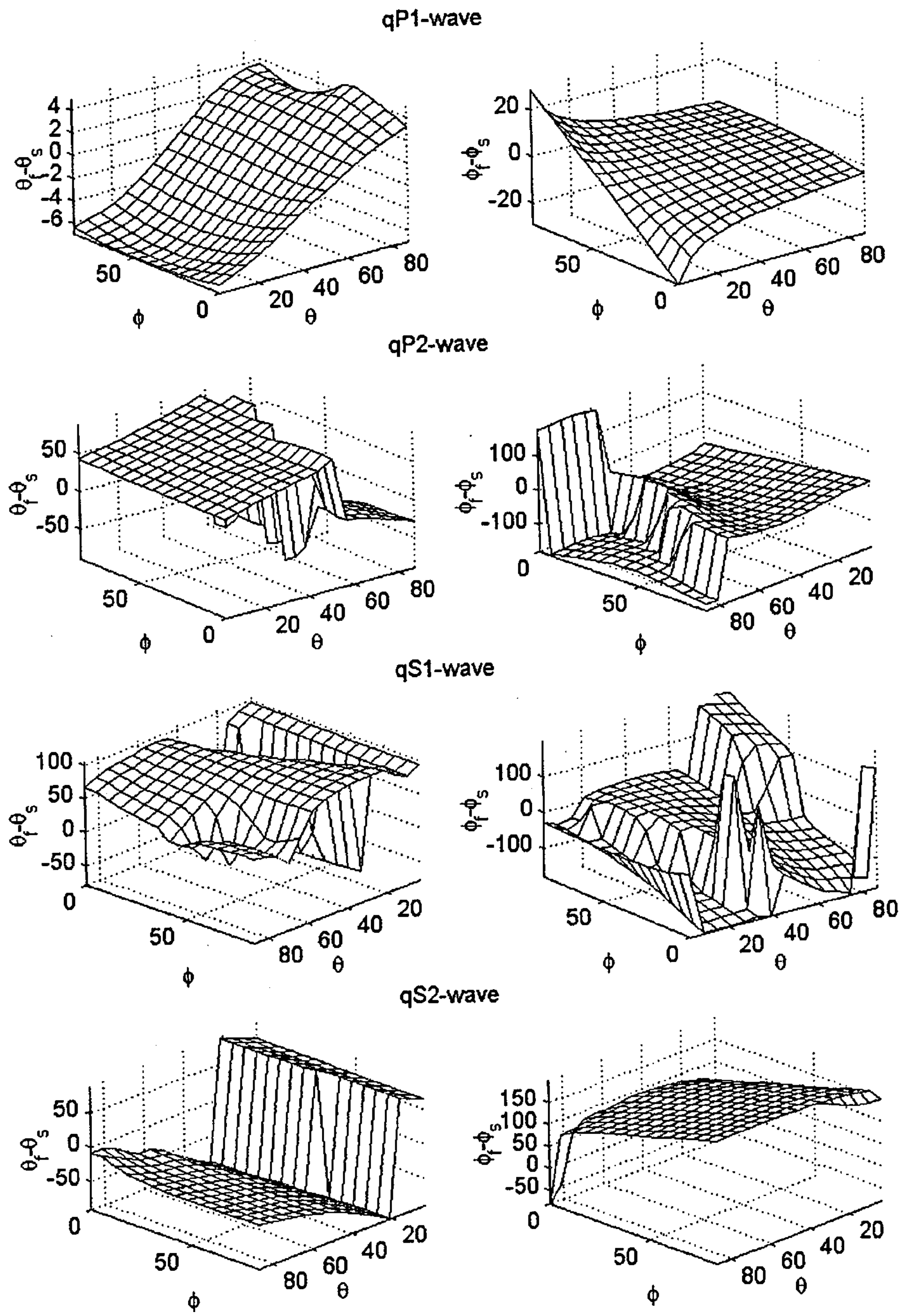

Figure 9. Variations of deviations of fluid particles from solid particles with phase direction (θ, ϕ): matrix porosity = 23 per cent; all angles in degrees.

7.2 Polarization

The numerical work is restricted to calculate the deviations of fluid and solid particles from phase direction. Keeping in view the practical importance of apparent polarizations, the deviations of fluid and solid particles from ray direction are also computed for all the four quasi-waves. The variations in the polarization of solid particles and fluid particles with phase direction are presented in Figs 5–8 .

Fig. 5 exhibits the deviations of vibrations of solid particle from the direction of phase propagation. For the qP1 wave the polar angle of solid particles deviates by nearly 5° whereas azimuth deviates up to 40°. Polar angle of solid particles for the qP2 wave deviates between 40° and 140°. This means that this motion is more transverse than longitudinal. The polar angle for solid particles in qS1 motion varies continuously with phase direction but its azimuth may suddenly reverse for some phase directions. The deviations of solid particles in qS2 motion is large but continuous. The deviations of the polarization from phase direction of fluid particles are plotted in Fig. 6. In qP1 motion, the fluid particles are polarized nearly along the phase direction. However, in the qP2 motion, the fluid particles move nearly along the phase direction for most of the phase directions but for some phase directions polar angles and azimuth can suffer sudden jumps of 90° and 180°,

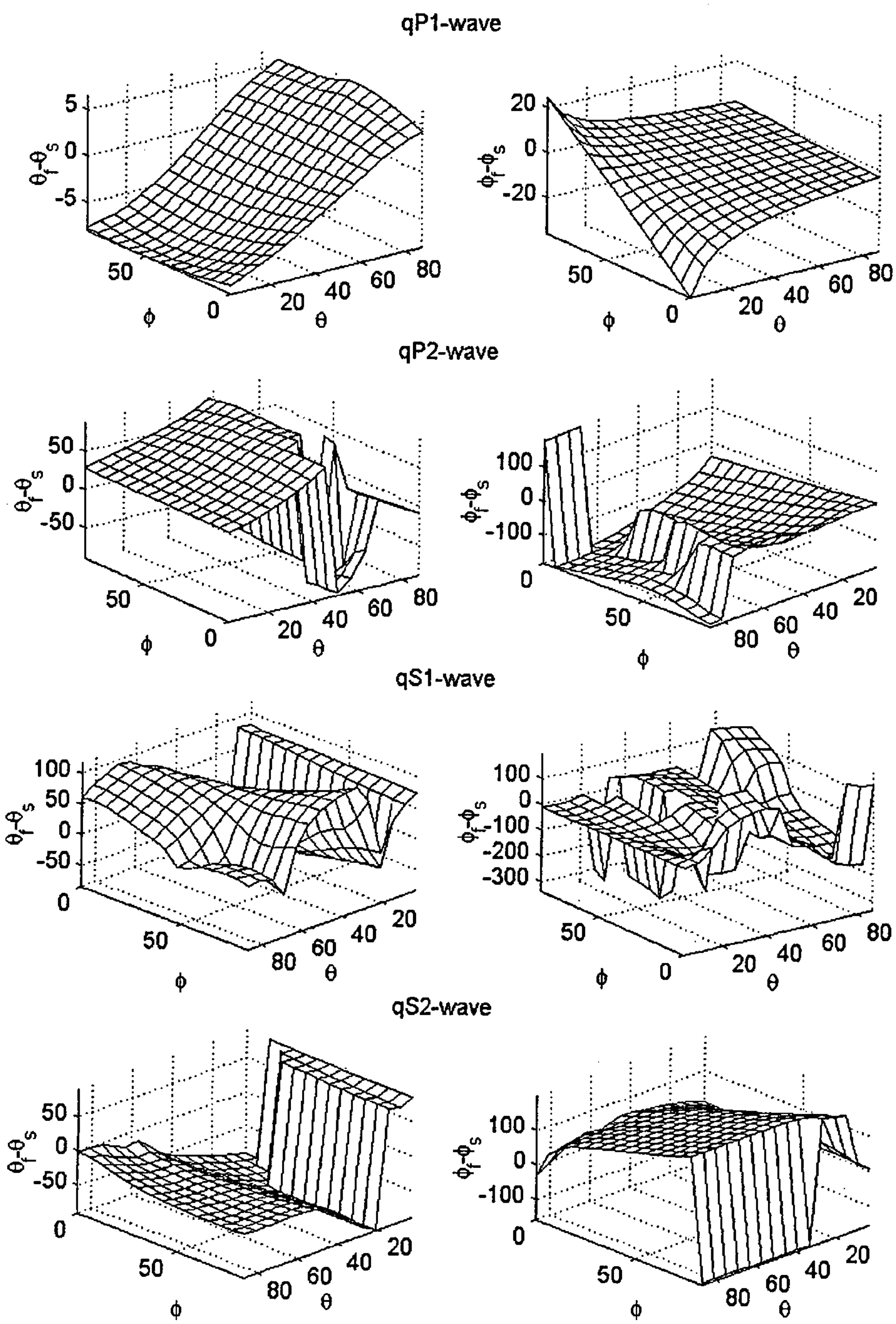

Figure 10. Variations of deviations of fluid particles from solid particles with phase direction (θ, ϕ): matrix porosity = 5 per cent; all angles in degrees.

respectively. The deviations of fluid particles from phase direction is also small in qS2 wave. This implies the smaller contribution of fluid particles in the qS2 wave. The motion of fluid particles in qS1 wave seems more turbulent than solid particles.

The variations of apparent polarizations of solid and fluid particles are presented in Figs 7 and 8. The apparent polarization of solid particles, in qP1 motion (Fig. 7) is much smaller as compared with the deviations from phase direction (Fig. 5). For the qP2, qS1 and qS1 waves, the polar angle in apparent deviations are nearly similar to its deviations from the phase direction. The azimuth deviations are also similar except in qP2 motion at larger θ. This implies that for these waves, the deviations of ray direction from phase direction are much smaller as compared with their apparent polarizations. Fig. 8 exhibits the apparent polarization of fluid particles. In qP1 motion, the apparent polarization of fluid particles is small but much larger as compared with their deviations from phase direction (Fig. 6). For other waves, the deviations in apparent polarizations are nearly similar to the deviations from phase direction.

In an anisotropic poroelastic medium, the motion of fluid particles may not be along (or parallel to) the motion of solid particles. The deviations of fluid particles from solid particles are calculated for two values (5 and 23 per cent) of matrix porosity and are plotted in Figs 9 and 10, respectively. These figures indicate the different behaviours of two constituents of a poroelastic solid. Changing the proportions of these two constituents changes the polarizations of the quasi-waves (particularly, the azimuth of the qS1 and qS2 waves). This implies that polarization difference of two constituents of a poroelastic solid may be diagnostic to the amount of fluid saturating the solid matrix.

The above discussion of numerical results is made for a particular model. So, it may not be useful to draw conclusions from such a discussion. The analytical expressions derived in this work may be used to improve the mathematical models of anisotropic wave propagation in poroelastic media. The researchers in this field would prefer to use these expressions to improve their models when interpreting the complex data. Polarization anomalies are a diagnostic feature of wave propagation in a cracked or anisotropic solid, and the present study certainly supplements this diagnosis. The difference in polarizations of two constituents of a poroelastic solid may help to study the type of anisotropy and amount of matrix porosity in porous rocks. This study may be helpful to the prospecting seismologists working for improved oil recovery. The expressions derived for the propagation of quasi-waves can, further, be used to derive the characteristic equations for surface waves and to study the reflection/refraction/scattering phenomena in general anisotropic poroelastic media.

REFERENCES

Aki, K. & Richards, P.G., 1980. *Quantitative Seismology, Theory and Methods*, Freeman, New York.

Auriault, J.L., 1980. Dynamic behaviour of a porous medium saturated by a Newtonian fluid, *Int. J. Eng. Sci.,* **18,** 775–785.

Bamford, D. & Nunn, K.R., 1979. *In-situ* seismic measurements of crack anisotropy in the carboniferous limestone of northwest England, *Geophys. Prospect.,* **27,** 322–338.

Ben-Menahem, A. & Sena, A.G., 1990. Seismic source theory in stratified anisotropic media, *J. geophys. Res.,* **95,** 15 395–15 427.

Ben-Menahem, A. & Singh, S.J., 1981. *Seismic Waves and Sources*, Springer, New York.

Ben-Menahem, A., Gibson, R.L., Jr & Sena, A.G., 1991. Green's tensor and radiation patterns of point sources in general anisotropic inhomogeneous elastic media, *Geophys. J. Int.,* **107,** 297–308.

Biot, M.A., 1955. Theory of elasticity and consolidation for a porous anisotropic solid, *J. Appl. Phys.,* **26,** 182–185.

Biot, M.A., 1956. The theory of propagation of elastic waves in a fluid-saturated porous solid, I. Low-frequency range, II. Higher frequency range, *J. acoust. Soc. Am.,* **28,** 168–191.

Burridge, R. & Keller, J.B., 1981. Poroelasticity equations derived from microstructure, *J. acoust. Soc. Am.,* **70,** 1140–1146.

Corrigan, D., 1989. Anisotropy in exploration seismology: experimental evidence, paper presented at Mass Institute of Technology, Cambridge.

Crampin, S., 1985. Evaluation of anisotropy by shear wave splitting, *Geophysics,* **50,** 383–411.

Crampin, S., 1994. The fracture criticality of crustal rocks, *Geophys. J. Int.,* **118,** 428–438.

Crampin, S. & McGonigle, R., 1981. The variations of delays in stress induced polarisation anomalies, *Geophys. J. R. astr. Soc.,* **64,** 115–131.

Fryer, G.J. & Frazer, N.L., 1987. Seismic waves in stratified anisotropic media—II. Elastodynamic eigensolutions for some anisotropic systems, *Geophys. J. R. astr. Soc.,* **91,** 73–101.

Gupta, I.N., 1973. Premonitory variations in *S*-wave anisotropy before earthquakes in Nevada, *Science,* **182,** 1129–1132.

Helbig, K., 1984. Transverse isotropy in exploration seismics, *Geophys. J. R. astr. Soc.,* **76,** 79–88.

Hudson, J.A., 1980. Overall properties of a cracked solid, *Math. Proc. Camb. Phil. Soc.,* **64,** 133–150.

Hudson, J.A., Liu, E. & Crampin, S., 1996. The mechanical properties of materials with interconnected cracks and pores, *Geophys. J. Int.,* **124,** 105–112.

Kerner, C., Dyer, B. & Worthington, M., 1989. Wave propagation in a vertical transversely isotropic medium: field experiment and model study, *Geophys. J. R. astr. Soc.,* **97,** 295–309.

Leary, P.C. & Henyey, T.L., 1985. Anisotropy and fracture zones about a geothermal well from *P*-wave velocity profiles, *Geophysics,* **50,** 25–36.

Leary, P.C., Li, Y.-G. & Aki, K., 1987. Observations and modeling of fault-zone fracture seismic anisotropy, *I*, *P*, *SV* and *SH* travel times, *Geophys. J. R. astr. Soc.,* **81,** 461–484.

Levy, T., 1979. Propagation of waves in a fluid-saturated porous elastic solid, *Int. J. Eng. Sci.,* **17,** 1005–1014.

Meadows, M.A. & Winterstein, D.F., 1994. Seismic detection of a hydraulic fracture from shear-wave VSP data at Lost Hills Field, California, *Geophysics,* **59,** 11–26.

Mengi, Y. & McNiven, H.D., 1978. Propagation and decay of waves in porous media, *J. acoust. Soc. Am.,* **64,** 1125–1131.

Rasolofosaon, P.N.J. & Zinszner, B.E., 2002. Comparison between permeability anisotropy and elasticity anisotropy of reservoir rocks, *Geophysics,* **67,** 230–240.

Rathore, J.S., Fjaer, E., Holt, R.M. & Renlie, L., 1995. *P*- and *S*-wave anisotropy of a synthetic sandstone with controlled crack geometry, *Geophys. Prospect.,* **43,** 711–728.

Sayers, C.M., 1988. Inversions of ultrasonic wave velocity measurements to obtain the microcrack orientation distribution function in rocks, *Ultrasonics,* **26,** 73–77.

Schmitt, D.P., 1989. Acoustical multipole logging in transversely isotropic poroelastic formations, *J. acoust. Soc. Am.,* **86,** 2397–2421.

Sharma, M.D., 1991. Wave propagation in liquid-saturated porous solids, *J. acoust. Soc. Am.,* **90,** 1068–1073.

Sharma, M.D., 1996. Surface wave propagation in a cracked poroelastic half-space lying under a uniform layer of fluid, *Geophys. J. Int.,* **127,** 31–39.

Sharma, M.D., 2002. Group velocity along general direction in a general anisotropic medium, *Int. J. Solids Struct.,* **39,** 3277–3288.
Thomsen, L., 1986. Weak elastic anisotropy, *Geophysics,* **51,** 1951–1966.
Thomsen, L., 1995. Elastic anisotropy due to aligned cracks in a porous rock, *Geophys. Prospect.,* **43,** 805–829.
Tod, S.R., 2002. The effects of stress and fluid pressure on the anisotropy of interconnected cracks, *Geophys. J. Int.,* **149,** 149–156.

APPENDIX A: ISOTROPIC PHASE VELOCITIES

To ensure the correctness of the expressions derived for anisotropic poroelasticity, in the text, eq. (10) is reduced to calculate the wave velocities in an isotropic poroelastic solid. Let a_{11}, a_{66}, Q and R be the four elastic constants that represent isotropic poroelasticity. Considering it a special case of anisotropic poroelasticity, we have

$$m_{ij} = Q\delta_{ij}; \qquad X_i = \frac{Q}{R}n_i; \qquad Z_{ij} = a_{66}\delta_{ij} + (a_{11} - a_{66})n_i n_j; \qquad Y_{ij} = \left(r_{12}^2 - 2r_{12}Q\big/R\right) n_i n_j;$$

$$P_{ij} = a_{66}\delta_{ij} + \left(a_{11} - a_{66} + r_{12}^2 - 2r_{12}Q/R\right) n_i n_j; \qquad Q_{ij} = \left(r_{12}^2 - 2r_{12}Q/R + Q^2/R^2\right) n_i n_j;$$

$$T_1 = -a_{66}\left(2a_{11} + a_{66} + 2r_{12}^2 - 4r_{12}Q/R\right); \qquad T_2 = -2a_{66}\left(r_{12}^2 - 2r_{12}Q/R + Q^2/R^2\right);$$

$$T_3 = a_{66}^2\left(r_{12}^2 - 2r_{12}Q/R + Q^2/R^2\right); \qquad \det\{P_{ij}\} = a_{66}^2\left(a_{11} + r_{12}^2 - 2r_{12}Q/R\right);$$

$$c_1 = (a_{11} + 2a_{66} + r_{11} - 2r_{12}Q/R)/g_0;$$

$$c_2 = \left(a_{11} - Q^2/R^2\right)/g_0 + a_{66}[2(a_{11} + r_{11} - 2r_{12}Q/R) + a_{66}]/g_0^2;$$

$$c_3 = 2a_{66}\left(a_{11} - Q^2/R^2\right)/g_0^2 + a_{66}^2(a_{11} + r_{11} - 2r_{12}Q/R)/g_0^3;$$

$$c_4 = a_{66}^2\left(a_{11} - Q^2/R^2\right)/g_0^3.$$

Eq. (10) reduces to

$$\left[h^2 - (a_{11} + r_{11} - 2r_{12}Q/R)h/g_0 + \left(a_{11} - Q^2/R^2\right)/g_0\right](h - a_{66}/g_0)^2 = 0,$$

and its roots are given by

$$h_1 = \left[(a_{11} + r_{11} - 2r_{12}Q/R) + \sqrt{(a_{11} + r_{11} - 2r_{12}Q/R)^2 - 4(a_{11} - Q^2/R^2}\right]/(2g_0),$$

$$h_2 = \left[(a_{11} + r_{11} - 2r_{12}Q/R) - \sqrt{(a_{11} + r_{11} - 2r_{12}Q/R)^2 - 4(a_{11} - Q^2/R^2}\right]/(2g_0),$$

$$h_3 = h_4 = a_{66}/g_0$$

Here, h_1, h_2 represent P_f, P_s waves, respectively, and $h_3 = h_4$ represent the shear wave. The wave velocities $v_j = \sqrt{Rh_j/\rho_{22}}$, ($j = 1, 2, 3$), are the same as defined in Biot's theory.

APPENDIX B: ISOTROPIC POLARIZATIONS

The polarizations expressions are reduced to become applicable to an isotropic poroelastic medium. a_{11}, a_{66}, Q and R are the four elastic constants for the isotropic poroelastic medium. With the relations $P_{ij} = a_{66}\delta_{ij} + (a_{11} - a_{66} + r_{12}^2 - 2r_{12}Q/R)n_i n_j$ and $Q_{ij} = (r_{12}^2 - 2r_{12}Q/R + Q^2/R^2)n_i n_j$, the three sets of Γ_i are given by

(i) $\Gamma_1 = -\Lambda_2 n_1 n_3, \qquad \Gamma_2 = -\Lambda_2 n_2 n_3, \qquad \Gamma_3 = \Lambda_1 + \Lambda_2\left(1 - n_3^2\right);$

(ii) $\Gamma_1 = -\Lambda_2 n_1 n_2, \qquad \Gamma_2 = \Lambda_1 + \Lambda_2\left(1 - n_2^2\right), \qquad \Gamma_3 = -\Lambda_2 n_2 n_3;$

(iii) $\Gamma_1 = \Lambda_1 + \Lambda_2\left(1 - n_1^2\right), \qquad \Gamma_2 = -\Lambda_2 n_1 n_2, \qquad \Gamma_3 = -\Lambda_2 n_1 n_3;$

where $\Lambda_1 = a_{66} - g_0 h$ and $\Lambda_2 = [(a_{11} - a_{66} + r_{12}^2 - 2r_{12}Q/R)h - a_{11} + a_{66} + Q^2/R^2]/(h - 1)$.

For the compressional waves (i.e. P_f and P_s)

$$g_0 h^2 - (a_{11} + r_{11} - 2r_{12}Q/R)h + \left(a_{11} - q^2/R^2\right) = 0;$$

and this is equivalent to $\Lambda_1 + \Lambda_2 = 0$. The polarizations of solid particles are then given by

$$\frac{u_1}{n_1} = \frac{u_2}{n_2} = \frac{u_3}{n_3}.$$

Polarizations of fluid particles $U_i = \frac{1}{h-1}(\frac{Q}{R} - r_{12}h)u_i$ implies that fluid particles moves parallel to the motion of solid particles.

For shear waves, $h = a_{66}/g_0$ implies $\Lambda_1 = 0$ and the polarization vectors for solid particles are

$$\left(1 - n_1^2, -n_1 n_2, -n_1 n_3\right); \qquad \left(-n_1 n_2, 1 - n_2^2, -n_2 n_3\right); \qquad \left(-n_1 n_3, -n_2 n_3, 1 - n_3^2\right).$$

The polarizations of fluid particles and solid particles are related through $U_i = -r_{12}u_i$, ($i = 1, 2, 3$). This polarization of fluid particles is due to the fluid-solid coupling which is represented by the dynamic constant ρ_{12}. According to Biot's theory (1956b), the value of r_{12} is, generally, very small and negative. This implies that during the propagation of shear waves, the fluid particles move with the solid particles but their motion is very small.

APPENDIX C: DISSIPATION IN POROUS SOLID

The expressions derived in text and previous appendices are valid for non-dissipative poroelastic solids (i.e. interstitial fluid is a non-viscous one). By transforming the dynamic constants

$$\rho_{11} \to \rho_{11} + \imath \frac{b}{\omega};$$
$$\rho_{22} \to \rho_{22} + \imath \frac{b}{\omega};$$
$$\rho_{12} \to \rho_{12} - \imath \frac{b}{\omega};$$

the non-dissipative poroelastic solid can be generalized to a dissipative poroelastic solid. Following Biot (1956a), we have the dissipation function

$$b = \frac{\nu}{\chi} f^2,$$

where ν is fluid viscosity, χ is permeability and f is porosity. This expression is valid for the low-frequency range, where the flow in the pores is of Poiseuille type. For higher frequencies, a correction factor is applied to the viscosity ν, replacing it by $\nu\Phi$, where Φ is a complex function of frequency. The coefficients of characteristic eq (10) will become complex. The velocities v_j, ($j = 1, 2, 3, 4$), derived from the roots of this equation, will also be complex.