Geophys. J. Int. (2004) **157,** 947–958 doi: 10.1111/j.1365-246X.2004.02226.x

3-D wave propagation in a general anisotropic poroelastic medium: reflection and refraction at an interface with fluid

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Accepted 2003 December 17. Received 2003 October 7; in original form 2003 March 25

SUMMARY

Reflection/refraction process is studied to calculate the energy distribution and transmission in a general anisotropic poroelastic solid half-space in contact with a fluid half-space. Biot's theory is used to study the propagation of plane harmonic waves in an anisotropic fluid-saturated porous solid. Snell's law for reflection/refraction at an interface between fluid and anisotropic poroelastic solid is calculated for the wave propagation in three dimensions. From the velocity and direction of incident wave in the fluid medium, Snell's law is used to find the phase velocities and phase directions of all the four refracted waves. The phase velocity and phase direction, thus, obtained are used to calculate the group velocity and ray direction of each of the refracted waves. The energy of the incident wave is distributed among one reflected wave and four quasi-waves refracted to the anisotropic poroelastic medium. Energy transmission with each of the reflected/refracted waves is studied along with their directions of travel in 3-D space. The variations of energy partition with the direction of the incident wave are computed for a particular model. The effect of azimuthal anisotropy on the partition and transmission of energy is observed.

Key words: energy travel, general anisotropy, poroelastic solids, reflection seismology, refraction seismology.

1 INTRODUCTION

Dynamic behaviour of fluid-saturated porous media is attracting considerable attention as a result of its importance in earthquake engineering, oil exploration, soil dynamics and hydrology. The poroelastic equations formulated by Biot (1956) have long been regarded as standard and have formed the basis for solving wave-propagation problems in poroelasticity. Biot (1955) presented the stress–strain relations for an anisotropic poroelastic solid. Anisotropy in porous solids may be the result of propagation through distribution of aligned cracks, microcracks and preferentially-oriented pore space. Occasionally, anisotropy may be the result of some other phenomena, such as rock foliation or crystal alignment. Following Biot's theory, Schmitt (1989) and Sharma (1991) studied the wave propagation in transversely isotropic poroelastic solids. Thomsen (1995) related the anisotropy to crack parameters in a porous rock and suggested that amount and type of anisotropy in the porous rocks depend upon the crack density, crack shape, stiffness of interstitial fluid, equant porosity, frequency, fluid pressure and flow between cracks and pores. This work was supported by the experimental study of anisotropy of sandstone with controlled crack geometry by Rathore *et al.* (1995). Sharma (1996) discussed the coexistence of cracks and pores and its effect on surface wave propagation. Hudson *et al.* (1996) studied the effect of connection between cracks and of small-scale porosity within the solid material on the overall elastic properties of cracked solids.

The study of anisotropic elasticity is also important for understanding the mechanical behaviour of composite materials (Braga 1990; Fan & Hwu 1998). Anisotropy in these materials results from the presence of crystals of particular symmetry or periodic thin laminates. In the last two decades, the applications of acoustic microscopy and fibre-reinforced composites have initiated interest in wave propagation in layered anisotropic media (Braga 1990). The stress-induced anisotropy in granular media (Norris & Johnson 1997) represents an active area of current research activity.

As available in the literature, the analytical studies on anisotropic propagation restrict the motion to a fixed (symmetry or arbitrary) plane and, hence, solve a 2-D problem. The energy propagation in an anisotropic media is, in fact, a 3-D phenomenon. The presence of mineral orientations, microfracturing or thin layering or combinations of these in a material results in a general anisotropy of arbitrary symmetry. The absence of symmetry in the aligned microcracks or pore space, also, results in the anisotropy of general type. The wave motion restricted to a symmetry plane represents only a special case of general anisotropy. Study of propagation in any one plane (particularly a symmetry plane) may give no indication of its behaviour in the neighbouring directions. It is, usually, impossible to extrapolate from a special case of anisotropy to the general. This demands that anisotropic wave propagation must be studied in 3-D space and the anisotropy considered should be of general type.

The reflection and transmission of waves at a fluid/porous solid interface seems to an important study. Starting with Geertsma & Smit (1961), to Deresiewicz & Rice (1964), Wu *et al.* (1990), Santos *et al.* (1992), Albert (1993), and Cieszko & Kubik (1998), it is continued with Denneman *et al.* (2002). In this last study, the closed form expressions of reflection and transmission coefficients (Denneman *et al.* 2000; Denneman *et al.* 2001) are calculated for the interfaces of water with water-saturated and air-filled porous layers. The work presented here, also, studies the reflection and transmission at the fluid/porous solid interface, however, the porous solid is considered anisotopic with arbitrary symmetry. The wave propagation is studied in 3-D space. The expressions for phase velocities and polarizations (Appendix A) of four quasi-waves are extracted from Sharma (2004, referred to as Paper I hereafter). The method used in this work is a different one. It is, mathematically, more exploring and can be applied to study the poroelastic anisotropies of all kinds.

2 FIELD EQUATIONS OF ANISOTROPIC POROELASTICITY

Following Biot (1956), the governing equations for a fluid-saturated porous media, in the absence of body forces and dissipation, are

$$\sigma_{ij,j} = \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i;$$

$$\sigma_{,i} = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i. \qquad (1)$$

The u_i and U_i are the components of the average displacements for the solid and fluid phases, respectively. The dot notation is used to represent time (partial) derivative. Summation convention is valid for repeated indices, which can assume the values 1, 2 and 3. The comma (,) before an index represents partial space differentiation. ρ_{11}, ρ_{12} and ρ_{22} are the dynamic constants depending upon the porosity (f) of solid, fluid-solid coupling and densities of solid particles and interstitial fluid. In an anisotropic porous material, the constitutive equations for stresses in the solid phase (i.e. σ_{ij}) and fluid (i.e. σ) are

$$\sigma_{ij} = c_{ijkl}u_{k,l} + m_{ij}U_{k,k};$$

$$\sigma = m_{ij}u_{i,j} + RU_{k,k}. \qquad (2)$$

The coefficients c_{ijkl} $(= c_{klij} = c_{jikl})$, m_{ij} $(= m_{ji})$ and R are the 28 independent material constants of a linear porous material.

To seek the harmonic solution of eq. (1), for the propagation of plane waves, write

$$u_j = S_j \exp\left[\iota\omega\left(\frac{1}{v}n_k x_k - t\right)\right];$$

$$U_j = F_j \exp\left[\iota\omega\left(\frac{1}{v}n_k x_k - t\right)\right], \quad (j = 1, 2, 3), \qquad (3)$$

where, ω is frequency, v is phase velocity of a wave along the phase direction (n_1, n_2, n_3), and S_j and F_j are the polarizations of solid and fluid particles, respectively. These polarizations are related by $F_j = G_{jk}S_k$ with G_{jk} defined in Appendix A (eq. A14). The Christoffel equation is a system of three homogeneous equations, given by

$$W_{ij}S_j = 0, \quad (i = 1, 2, 3), \qquad (4)$$

where W_{ij}, the elements of square matrix of order 3, are as defined in Appendix A. Non-trivial solution of this system explains the propagation of four quasi-waves ($qP1$, $qP2$, $qS1$, $qS2$) in an anisotropic poroelastic medium. The phase velocities of these quasi-waves are given by $v_j = \sqrt{Rh_j/\rho_{22}}$, ($j = 1, 2, 3, 4$) with h_j as defined in Appendix A. These phase velocities depend upon the direction of phase propagation.

3 REFLECTION AND REFRACTION

According to the method used by Keith & Crampin (1977), the Christoffel equation is reduced to a polynomial that is solved for the vertical slowness values. The vertical slowness value for a wave represents the ratio of its (vertical) direction cosine to its phase velocity. Phase velocity being a function of phase direction is not obtained, exclusively, and, hence, Snell's law does not appear in the whole scene. The energy flux is used in finding the ray directions of reflected/refracted quasi-waves. Schmitt (1989) and Sharma (1991) used this method to study the wave propagation in transversely isotropic poroelastic solids. For general anisotropy, this method will require the solution of a polynomial of degree 8 to find the slowness values for all the waves propagating in an anisotropic poroelastic solid. This is certainly a difficult task, particularly for complex roots. A new method is proposed, which is nearly analogous to the process used to study reflection/refraction in isotropic media. The proposed process traverses through the phase directions, phase velocities of refracted quasi-waves and Snell's law at the fluid/anisotropic medium interface. The directional derivatives of phase velocities of quasi-waves can be obtained analytically and can be used in finding the group velocities and ray directions of transmitted quasi-waves. This method is explained as follows:

(i) The Christoffel equation is solved analytically to find the phase velocities (eq. A10) of all the quasi-waves in an anisotropic poroelastic medium.

(ii) The phase velocity of a quasi-wave depends upon its phase direction. The phase direction is obtained from Snell's law, which involves the phase velocity of the quasi-wave. This is a tricky situation. Here, the analytical expression for phase velocity of the quasi-wave manages

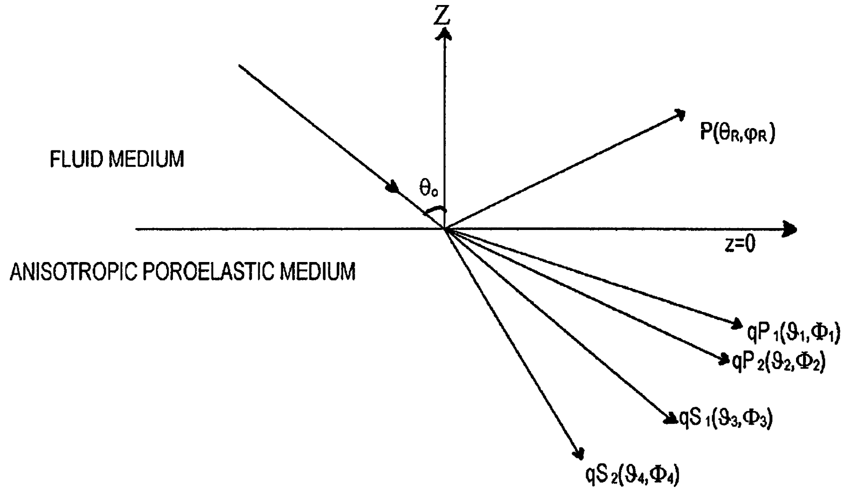

Figure 1. Geometry of the medium (rays represent phase directions).

the way out. Along with Snell's law (eq. 10), the Bisection method (or Newton's method) solves a non-linear equation (eq. 11) to find the phase direction of the quasi-wave for the given values of phase velocity and phase direction of the incident wave.

(iii) The phase direction obtained numerically from the non-linear equation, in step (ii), is used to calculate the phase velocity (Appendix A2) of the quasi-wave.

(iv) The Christoffel equation in the form of a polynomial (eq. A7) in phase velocity provides the expressions for directional derivatives of phase velocity (eq. 17). The phase direction from step (ii) and phase velocity from step (iii), along with its directional derivatives, are used to calculate the group (energy) velocity vector (eq. 14).

(v) Snell's law is used to calculate the vertical slowness for each of the four quasi-waves transmitted to porous medium from their phase directions and phase velocities.

3.1 Geometry of the medium

Consider a fluid half-space in contact with anisotropic poroelastic half-space along a plane interface. In a rectangular coordinate system (x, y, z), the plane $z = 0$ represents the interface between two half-spaces and the z-axis is pointing into the fluid, as shown in Fig. 1. An acoustic wave travels through the fluid and becomes incident at a point on the interface. In a spherical coordinate system centred at this point, let (θ_I, ϕ_o) be the direction of incident wave in 3-D space. According to the Fig. 1, $\theta_I = \pi - \theta_o$, where θ_o is the angle between incident ray and z-axis. This incident wave results in a reflected wave along (θ_R, ϕ_R) and four waves ($qP1$, $qP2$, $qS1$, $qS2$) refracted to the poroelastic medium. Rays in the poroelastic medium represent the phase directions (θ_j, ϕ_j), ($j = 1, 2, 3, 4$) of the four quasi waves.

3.2 Displacements

The displacement components in the fluid medium are written as

$$u_j^o = n_j \exp\left[\iota\omega\left(\frac{1}{v_o}n_k x_k - t\right)\right] + a_R n'_j \exp\left[\iota\omega\left(\frac{1}{v_o}n'_k x_k - t\right)\right], \quad (j = 1, 2, 3), \tag{5}$$

where v_o is the velocity of sound in fluid, $(n_1, n_2, n_3) = (\sin\theta_o\cos\phi_o, \sin\theta_o\sin\phi_o, -\cos\theta_o)$ and $(n'_1, n'_2, n'_3) = (\sin\theta_R\cos\phi_R, \sin\theta_R\sin\phi_R, \cos\theta_R)$.

The displacement components in the anisotropic poroelastic medium are expressed as

$$u_j = \sum_{m=1}^{4} a(m) S_j^{(m)} \exp\left[\iota\omega\left(\frac{1}{v_m}n_k^{(m)} x_k - t\right)\right];$$

$$U_j = \sum_{m=1}^{4} a(m) F_j^{(m)} \exp\left[\iota\omega\left(\frac{1}{v_m}n_k^{(m)} x_k - t\right)\right]; \quad (j = 1, 2, 3), \tag{6}$$

where $[n_1^{(m)}, n_2^{(m)}, n_3^{(m)}] = (\sin\theta_m\cos\phi_m, \sin\theta_m\sin\phi_m, \cos\theta_m)$ represents the phase direction of quasi-wave m. The $a(m)$ are relative excitation factors.

3.3 Boundary conditions

Following Deresiewicz & Skalak (1963), the boundary conditions at the interface between a fluid medium and a poroelastic medium are the continuity of stresses of fluid and solid constituents. Continuity of normal components of velocity are also considered. In this problem, the following are the five appropriate boundary conditions that are required to be satisfied at the plane $z = 0$:

$$\sigma_{zx} = 0,$$

$$\sigma_{zy} = 0,$$

$$\sigma_{zz} = (1 - f)K^o u^o_{j,j},$$

$$\sigma = f K^o u^o_{j,j},$$

$$(1 - f)\dot{u}_3 + f\dot{U}_3 = \dot{u}^o_3, \tag{7}$$

where, K^o is the bulk modulus of the fluid medium. σ represents the stress components in poroelastic medium and eq. (2) relates them to the displacement components (u_i, U_i). Satisfying the above boundary conditions with the displacements defined in last section yields a system of five linear inhomogeneous equations in $a(1)$, $a(2)$, $a(3)$, $a(4)$ and a_R. These equations are given by

$$\sum_{m=1}^{4} J_i^{(m)} \frac{R}{v_m} a(m) - (1 - f)\frac{K^o}{v_o}\delta_{i3} a_R = (1 - f)\frac{K^o}{v_o}\delta_{i3}, \quad (i = 1, 2, 3);$$

$$\sum_{m=1}^{4} J_4^{(m)} \frac{1}{v_m} a(m) - f\frac{K^o}{v_o} a_R = f\frac{K^o}{v_o},$$

$$\sum_{m=1}^{4} [(1 - f)S_3^{(m)} + f F_3^{(m)}]a(m) - n'_3 a_R = n_3, \tag{8}$$

where,

$$J_i^{(m)} = \sum_{k=1}^{3} \left[a_{lk} S_k^{(m)} + \frac{m_{i3}}{R} F_k^{(m)}\right] n_k^{(m)} + a_{l4}\left[S_2^{(m)} n_3^{(m)} + S_3^{(m)} n_2^{(m)}\right]$$
$$+ a_{l5}\left[S_1^{(m)} n_3^{(m)} + S_3^{(m)} n_1^{(m)}\right] + a_{l6}\left[S_1^{(m)} n_2^{(m)} + S_2^{(m)} n_1^{(m)}\right], \quad l = 6 - i; \quad (i = 1, 2, 3),$$

$$J_4^{(m)} = m_{kl} S_k^{(m)} n_l^{(m)} + R F_k^{(m)} n_k^{(m)}. \tag{9}$$

3.4 Snell's law

In order to solve the system of eq. (8) for $a(m)$ and a_R, the values of $n_j^{(m)}$ and v_m are required for a given direction (n_1, n_2, n_3) and velocity (v_o) of the incident wave. The continuity in boundary conditions require the identical phase of all the waves at the interface $z = 0$. The Snell's law in three dimensions is, then, explained by

$$\frac{n_i}{v_o} = \frac{n'_i}{v_o} = \frac{n_i^{(m)}}{v_m}, (m = 1, 2, 3, 4); \quad (i = 1, 2). \tag{10}$$

Following are some interesting points drawn from the Snell's law:

(i) $n'_2/n'_1 = n_2/n_1 = n_2^{(m)}/n_1^{(m)}$, $(m = 1, 2, 3, 4)$ imply that $\phi_m = \phi_R = \phi_o$. This means that the phase directions of all the reflected and refracted waves lie in the same vertical plane. So, in the case of azimuth isotropy, the velocity and direction of energy propagation will confine to this plane only. Hence, in an anisotropic medium with anisotropy up to azimuthal isotropy, the study of wave propagation in a plane is sufficient to explain the reflection/refraction phenomenon. However, the presence of azimuthal anisotropy demands that the wave propagation needs to be studied in three dimensions.

(ii) The polar angle of the wave reflected into the fluid is given by $\theta_R = \pi - \theta_I = \theta_o$.

(iii) The phase velocity v_m of the quasi-wave m in a poroelastic medium depends upon its phase direction (θ_m, ϕ_o). Using Snell's law and the condition $n_k^{(m)} n_k^{(m)} = 1$, an equation,

$$h_m \sin^2\theta_o - \left(\rho_{22} v_o^2/R\right)\sin^2\theta_m = 0, \tag{11}$$

is obtained which relates θ_m and v_m. An expression of $h_m = \rho_{22} v_m^2/R$ as a function of θ_m can be obtained from the Appendix A. Along with this expression and eq. (11), Bisection method (or Newton's method) will be able to derive the value of θ_m for any given value of θ_o. This value of θ_m is used to calculate the phase velocity v_m and, hence, partition, velocity and direction of energy.

(iv) It may be noted that polar angles θ_m, $(m = 1, 2, 3, 4)$ of quasi-waves are derived from the polar angle of incident wave. As the incident wave reaches the critical angle for any of the refracted waves, the refracted wave propagates along the interface. For the concerned refracted wave, the phase direction is given by $\theta_m = \pi/2$ and the phase velocity is given by $v_m^c = v_m(\pi/2, \phi_o)$. Hence, for the post-critical incidence, the concerned refracted wave has the constant phase velocity, i.e. v_m^c.

(v) Snell's law is modified to $\sin\theta_I/v_o = \sin\theta_m/v_m^c$. The critical angle for refracted quasi-wave m is then given by $\sin^{-1}(v_o/v_m^c)$. To calculate the energy partition for post-critical incidence, the vertical slowness is given by $n_3^{(m)}/v_m^c$, where $n_3^{(m)} = \cos\theta_m = \sqrt{[1 - (\sin\theta_I v_m^c/v_o)^2]}$ is an imaginary value.

3.5 Energy ratios

Distribution of energy between different reflected and refracted waves is considered across a surface element of unit area at the plane $z = 0$. Following Achenbach (1973), the scalar product of surface traction and particle velocity per unit area, denoted by P^*, represents the rate at which the energy is communicated per unit area of the surface. The time average of P^* over a period, denoted by $\langle P^*\rangle$, represents the average energy transmission per unit surface area per unit time. If z-axis is the outer normal to the surface, then, in the fluid medium, the average energy flux of incident and reflected waves are $\langle P_I^*\rangle = -0.5\omega^2 K^o n_3/v_o$ and $\langle P_R^*\rangle = -0.5\omega^2|a_R|^2 K^o n_3'/v_o$, respectively. Taking into account the energy transmitted to the fluid portion of poroelastic solid, the average energy carried by the quasi-wave m is

$$\left\langle P_m^*\right\rangle = -0.5\omega^2|a(m)|^2 Re\left[\sum_{k=1}^{3} RJ_k^{(m)}\bar{S}_k^{(m)} + J_4^{(m)}\bar{F}_3^{(m)}\right] \Big/ v_m, \quad (m = 1, 2, 3, 4). \tag{12}$$

The energy ratios of reflected and refracted waves [i.e. $E_R = \langle P_R^*\rangle/\langle P_I^*\rangle$ and $E_m = \langle P_m^*\rangle/\langle P_I^*\rangle$, $(m = 1, 2, 3, 4)$] ensure the conservation of energy by satisfying the relation $\sum_{m=1}^{4} E_m - E_R = 1$.

At incidence beyond the critical angle for a refracted wave, the waves become inhomogeneous and, hence, involve the concept of interaction energy. Borcherdt (1977) explained the existence of interaction energy for the reflection and refraction of *SH* waves. To ensure energy conservation, Ainslie & Burns (1995) have also explained some derivations involving interference-energy or complex-energy ratios. In the present problem, an energy matrix $E_{jk} = \langle P_{jk}^*\rangle\,/\langle P_I^*\rangle$; $(j, k = 1, 2, 3, 4)$, is defined to calculate the interaction energy among the four quasi-waves in poroelastic medium. The energy fluxes are defined as

$$\left\langle P_{jk}^*\right\rangle = -0.5\omega^2 Re\left\{\frac{1}{v_j}\left[\sum_{i=1}^{3} RJ_i^{(j)}a(j)\bar{S}_i^{(k)}\bar{a}(k) + J_4^{(j)}a(j)\bar{F}_3^{(k)}\bar{a}(k)\right]\right\}. \tag{13}$$

The sum of all the non-diagonal entries of this energy matrix gives the share of interaction energy for the refracted waves. In this case, the conservation of energy is given by the relation $\sum_{j=1}^{4}\sum_{k=1}^{4} E_{jk} - E_R = 1$. At incidence before all the critical angles, this energy matrix is nearly a skew symmetric one. Hence, a small interaction energy (only, for triclinic system) is observed even for incidence before critical angles. The diagonal entries of matrix E represent the energy ratios of the transmitted waves.

4 GROUP VELOCITY

In an anisotropic medium, the energy associated with a quasi-wave travels with its group velocity along a ray at an angle to its direction of phase propagation. In a spherical coordinate system let $v(\theta, \phi)$ define the phase velocity of a quasi-wave. According to the geometry considered in Fig. 1, such a wave travels in the vertical plane (i.e. $\phi = \phi_o$) making an angle θ with the z-axis. The components of group velocity, w_j, $(j = x, y, z)$, following Ben-Menahem & Sena (1990), are expressed as follows:

$$w_x/v = \cos\phi\sin\theta + \cos\phi\cos\theta T_\theta - \frac{\sin\phi}{\sin\theta}T_\phi;$$

$$w_y/v = \sin\phi\sin\theta + \sin\phi\cos\theta T_\theta + \frac{\cos\phi}{\sin\theta}T_\phi;$$

$$w_z/v = \cos\theta - \sin\theta T_\theta, \tag{14}$$

where, T_θ and T_ϕ are defined by,

$$T_k = \frac{1}{v}(v)_{,k} = \frac{1}{2h}h_{,k}; \quad (k = \theta, \phi). \tag{15}$$

The magnitude of the group velocity of the quasi-wave is

$$w = v\sqrt{1 + T_\theta^2 + \frac{1}{\sin^2\theta}T_\phi^2}, \tag{16}$$

and its ray direction, (θ_g, ϕ_g), is calculated from its components. The partial derivatives of h $(= \rho_{22}v^2/R)$, in eq. (15), are given by the relation

$$h' = \left(c_1'h^3 - c_2'h^2 + c_3'h - c_4'\right) / \left(4h^3 - 3c_1h^2 + 2c_2h - c_3\right). \tag{17}$$

The derivatives of coefficients (i.e. c_k', $k = 1, 2, 3, 4$) are derived analytically from the relations given in Appendix A.

5 NUMERICAL COMPUTATION AND DISCUSSION

The purpose of numerical computation is to study the partition of energy across the interface between two media. Anisotropy in a porous medium is a general one and it requires the propagation to be studied in three dimensions. The dolomite, a real crystalline rock, is considered as a general anisotropic poroelastic solid. Elastic matrix (GPa) for Dolomite, (Rasolofasaon & Zinszner 2002) is written as:

$c_{11} = 65.53 \quad c_{12} = 9.77 \quad c_{13} = 12.19 \quad c_{14} = 0.18 \quad c_{15} = -0.81 \quad c_{16} = 2.94;$

$c_{22} = 50.77 \quad c_{23} = 11.61 \quad c_{24} = -0.09 \quad c_{25} = -0.50 \quad c_{26} = -0.19;$

$c_{33} = 60.11 \quad c_{34} = -1.61 \quad c_{35} = 1.78 \quad c_{36} = 0.84;$

$c_{44} = 23.51 \quad c_{45} = 1.49 \quad c_{46} = -1.17 \quad c_{55} = 24.57 \quad c_{56} = 0.26 \quad c_{66} = 20.21.$

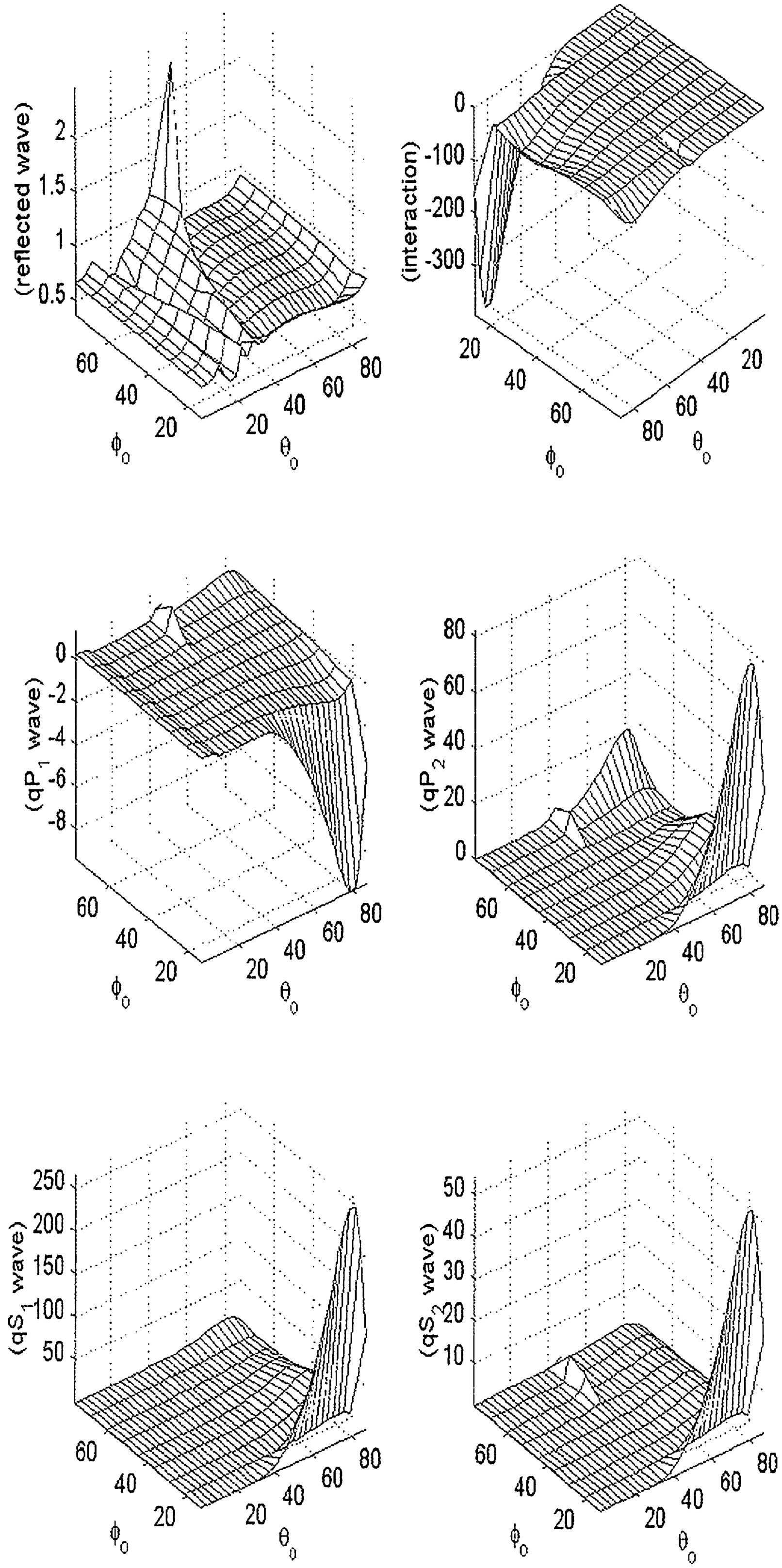

Figure 2. Variations of energy ratios with the direction of incidence (θ_o, ϕ_o; angles in degrees).

The density is 2423 kg m^{-3}. The values assumed for remaining elastic parameters (GPa) are (m_{11}, m_{22}, m_{33}, m_{12}, m_{13}, m_{23}) = (20, 21, 19, 1, 2, 2.5); $R = 15$. Dynamic constants are derived for 23 per cent porosity in a solid of density 2423 kg m^{-3} and containing a fluid of density 1000 kg m^{-3}. These are $\rho_{11} = 1770$ kg m^{-3}; $\rho_{12} = -10$ kg m^{-3}; $\rho_{22} = 235$ kg m^{-3}. For the fluid medium, $\rho = 1000$ kg m^{-3} is the density of water and $v_o = 1.463$ km s^{-1} is the velocity of sound in it.

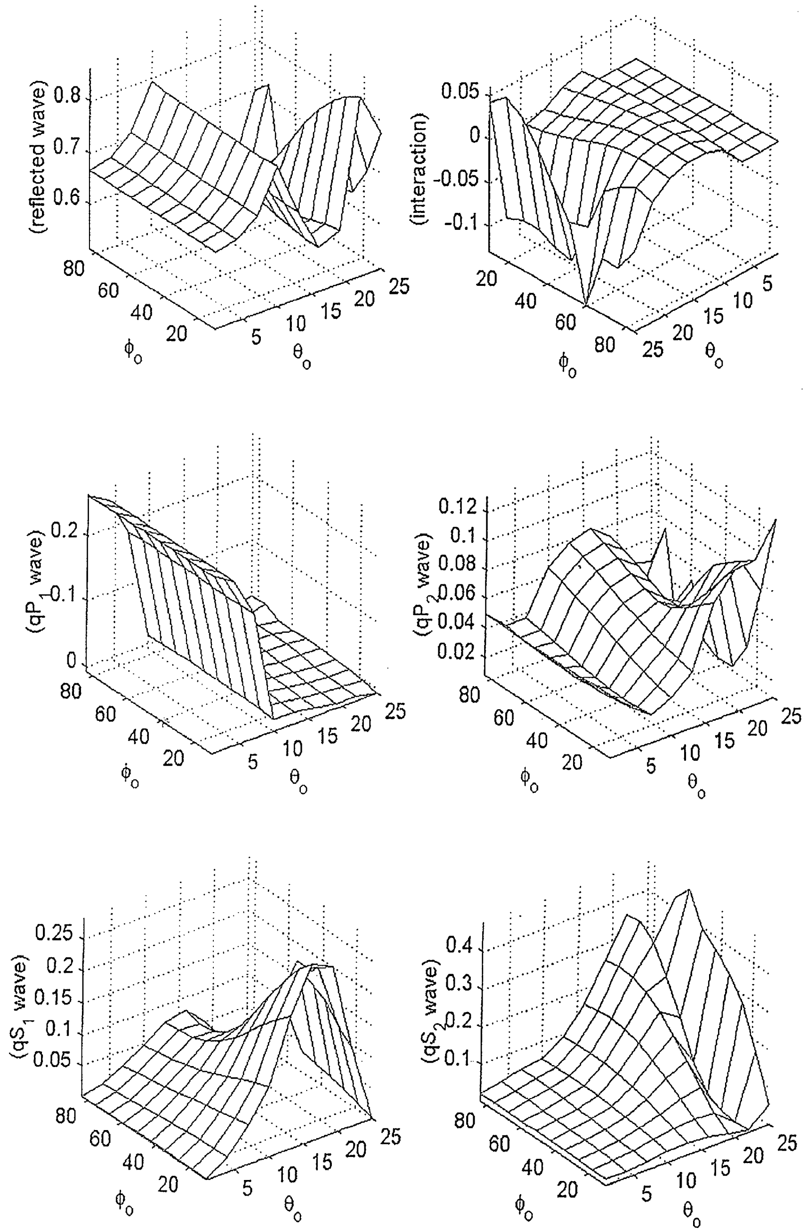

Figure 3. Variations of energy ratios with the direction of incidence (θ_o, ϕ_o; angles in degrees): incidence before critical angles of refracted waves.

Using the above numerical values, the variations of energy ratios in 3-D space are calculated and presented in Fig. 2. The polar angle θ_o (Fig. 1) and azimuth ϕ_o both take the values between 0° and 90°. As a result of the large difference between the phase velocities in water and poroelastic solid, the critical angles for all the refracted waves lie in the range of θ_o between 0° and 25°. After this range, the terms of energy matrix (*i.e.* E_{jk}) increases sharply. This gives larger variations in the energies and nullifies the variations of energies in the 0° to 25° range of incidence. Fig. 3. exhibits a clear picture of energy variations in this range. The variations of group velocities and ray directions of the waves transmitting in a poroelastic medium are exhibited in Figs 4 and 5. In case of post-critical incidence, the polar angle of phase propagation of the concerned refracted wave does not change. The change in the azimuth of the phase results in similar azimuthal variations and negligible polar angle variations in the group velocity and ray direction of the concerned wave. Keeping this in mind, the variations are plotted only for values of θ_o between 0° and 60°. Details are as follows.

As shown in Fig. 2, for values of θ_o beyond 25°, the energies of refracted waves and interaction energy becomes large enough to nullify the variations for smaller θ_o. This happens mainly for values of azimuth ϕ_o near 0° and 90° because the incidence is beyond the critical angles of refracted waves. The variations of energies for the pre-critical incidence are not readable. However, the conservation of energy holds well for any arbitrary angle of incidence.

In Fig. 3, as a result of the transmitting medium being much denser, the major part of the energy is reflected back. The energy transmitted to a poroelastic medium is carried mainly by the *qP*1 wave for incidence at a smaller polar angle. Energy shares of *qP*2, *qS*1 and *qS*2 waves increases for the incidence beyond the critical angle for *qP*1 wave. Interaction energy is not very significant. It is noted that the critical angles

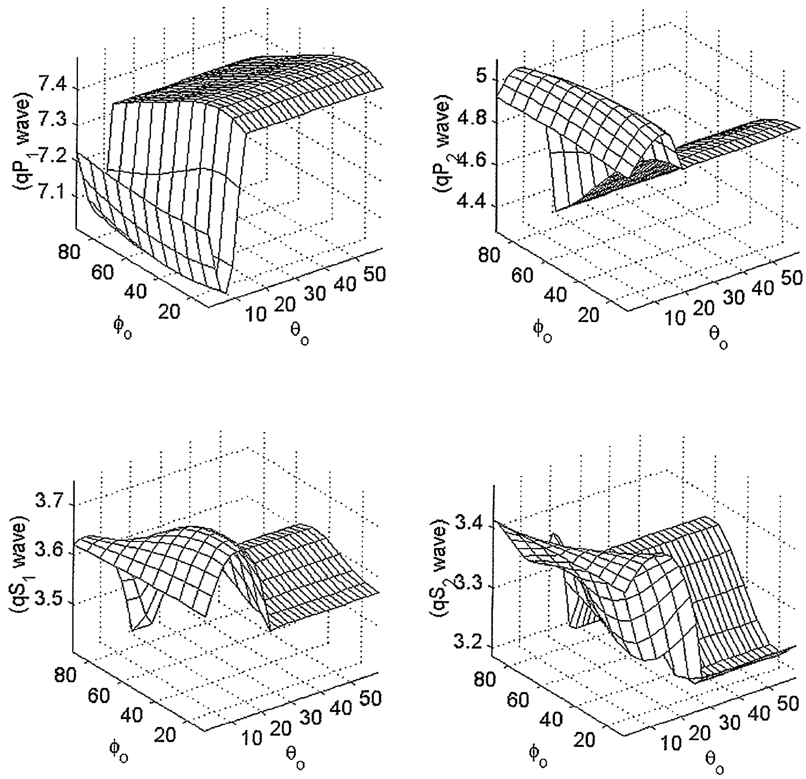

Figure 4. Variations of group velocity (km s^{-1}) with the direction of incidence (θ_o, ϕ_o; angles in degrees).

for various waves increases with the increase of azimuth (ϕ_o) of the incident wave. Also, for incidence at, near and after critical angles, the energy partition is strongly dependent upon the value of azimuth.

Fig. 4 shows that the group velocities of all four quasi-waves transmitting in the anisotropic poroelastic medium are strongly dependent upon both the polar angle and azimuth of the incident wave. The critical angle is dependent upon the azimuth of the incident wave. For each refracted wave, it is found to increase slightly with the increase in azimuth. The group velocities vary with azimuth even at incidence after critical angles. This indicates that the velocity of the interface waves are azimuth dependent.

In Fig. 5, the polar angle of a transmitted energy changes through 180° to 90° with the incidence changing from normal incidence to critical incidence. For post-critical incidence, the energy is confined to the planes cutting the fluid/porous interface at very smaller angles. For post-critical incidence, the azimuths of the refracted waves vary almost similarly to the variations in the azimuth of the incident wave. This implies that the refracted energy is, more or less, confined around the vertical plane of incidence. The azimuth variations of incident wave do influence the ray direction of each refracted wave at incidence before its critical angle.

The above numerical results are obtained for a realistic and, yet, particular model. These results may not qualify for generalization. The behaviour of these results on expected lines, certainly, verifies the correctness of the expressions derived in the work. The conservation of energy for incidence at an arbitrary angle in three dimensions certifies the applicability of the technique discussed in this work. The method presented calculates phase direction, phase velocity, ray direction, group velocity and critical angle of each of the refracted quasi-waves. Hence, it qualifies as a method to be more explanatory and transparent. The only numerical methods needed are the Gauss elimination method to solve a system of five linear equations and the Bisection method to find the polar angle of each of the four quasi-waves for a given direction of incident wave. Because finding the directional derivatives of phase velocities is not difficult, the Newton's method can replace Bisection method, if needed.

The purpose of this work is to study the energy transmission (i.e. group velocity and ray direction) in a general anisotropic poroelastic medium. Therefore, the present concern is the anisotropy with arbitrary symmetry. The effects of porous parameters on the propagation can be another important aspect to be considered in a later study. The work presented introduces a method to study the wave propagation in an anisotropic poroelastic medium with arbitrary symmetry. This method is equally applicable for wave propagation in non-porous anisotropic elastic solids. Such studies enable seismologists and structural engineers to use the improved models when interpreting their complex data. The anisotropic poroelasticity may be more useful in studying the dynamic behaviour of composite and granular materials. This work can, further, be used to study the propagation in multilayered models involving anisotropic poroelastic (APE) solids. The analytical studies create a space for further research work and, also, provide algorithms for numerical codes. Hence, the researchers in this field are likely to prefer to use the expressions derived in this work.

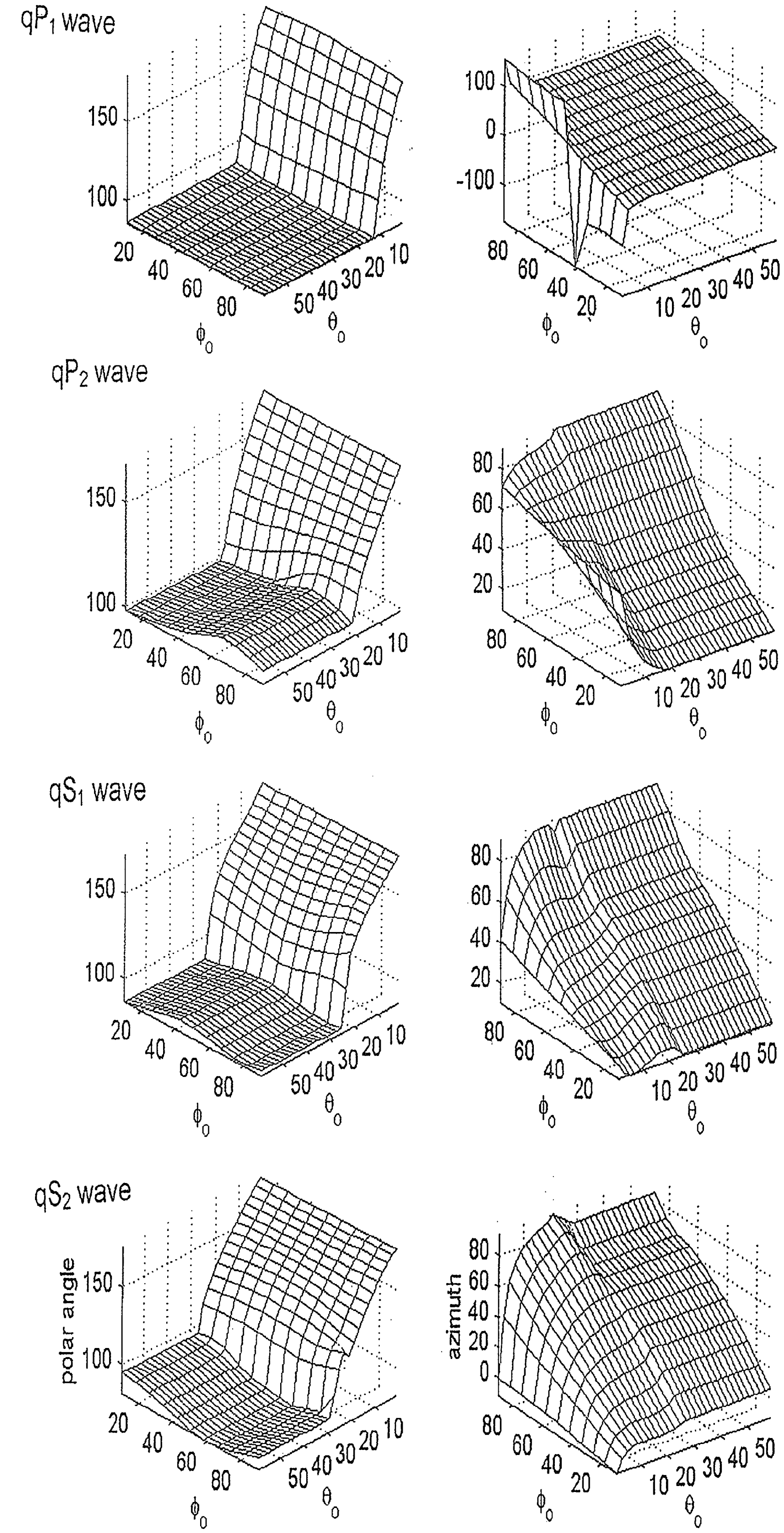

Figure 5. Variations of ray direction (polar angle, azimuth) with the direction of incidence (θ_o, ϕ_o; angles in degrees).

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APPENDIX A

A1 Christoffel equation

The Christoffel equation for wave propagation in an anisotropic poroelastic medium is given by

$$W_{ij} S_j = 0, \quad (i = 1, 2, 3), \tag{A1}$$

where S_j is polarization of solid particles. The elements of coefficient matrix are

$$W_{ij} = -g_o h \delta_{ij} + P_{ij} + \frac{1}{h-1} Q_{ij}, \quad (i, j = 1, 2, 3), \tag{A2}$$

where, δ_{ij} is Kronecker delta. P and Q, the square matrices of order 3, are defined as follows. Denote by the row matrix $N = (n_1, n_2, n_3)$, where n_j are the components of a unit vector normal to wave surface, the direction of phase propagation. Consider a general anisotropic poroelastic medium with elastic constants c_{ijkl} of the solid matrix represented by two-suffix notation, c_{ij}. Define, following Sharma (2002),

$$\alpha = NAN', \qquad \beta = NBN', \qquad \gamma = NCN',$$

$$\delta = NDN', \qquad \eta = NEN', \qquad \zeta = NFN', \tag{A3}$$

where N' denotes the transpose of row matrix N. A, B, C, D, E and F are square matrices of order 3 are defined as follows:

$$A = (a_{11}, a_{16}, a_{15}; a_{16}, a_{66}, a_{56}; a_{15}, a_{56}, a_{55}); \quad B = (a_{66}, a_{26}, a_{46}; a_{26}, a_{22}, a_{24}; a_{46}, a_{24}, a_{44});$$

$$C = (a_{55}, a_{45}, a_{35}; a_{45}, a_{44}, a_{34}; a_{35}, a_{34}, a_{33}); \quad D = (a_{16}, a_{12}, a_{14}; a_{66}, a_{26}, a_{46}; a_{56}, a_{25}, a_{45});$$

$$E = (a_{15}, a_{14}, a_{13}; a_{56}, a_{46}, a_{36}; a_{55}, a_{45}, a_{35}); \quad F = (a_{56}, a_{46}, a_{36}; a_{25}, a_{24}, a_{23}; a_{45}, a_{44}, a_{34}); \tag{A4}$$

where $a_{ij} = c_{ij}/R$. The matrix $P = Z + Y$ and matrix $Q = (Q_{ij})$ is defined by $Q_{ij} = X_i X_j + Y_{ij}$, where symmetric, square matrix

$$Z = (\alpha, \delta, \eta; \delta, \beta, \zeta; \eta, \zeta, \gamma), \tag{A5}$$

and the elements of symmetric matrix Y are

$$Y_{ij} = r_{12}^2 n_i n_j - r_{12}(n_i X_j + n_j X_i), \quad (i, j = 1, 2, 3). \tag{A6}$$

The elements of row matrix X are defined as $X_i = m_{ik}n_k/R$, $(i = 1, 2, 3)$. In eq. (A2), the variable $h = \rho_{22}v^2/R$ and $g_o = r_{11} - r_{12}^2$, where $r_{1j} = \rho_{1j}/\rho_{22}$, $(j = 1, 2)$.

A2 Phase velocities

The non-trivial solution of the Christoffel equation is ensured by a biquadratic equation

$$h^4 - c_1h^3 + c_2h^2 - c_3h + c_4 = 0. \quad \text{(A7)}$$

This equation has all its four roots [e.g. h_j, $(j = 1, 2, 3, 4)$] positive only if all its coefficients are positive. The phase velocities of the four quasi-waves, given by $v_j = \sqrt{(Rh_j/\rho_{22})}$, $(j = 1, 2, 3, 4)$, will be varying with the direction of phase propagation. These waves, represented by $j = 1, 2, 3$ and 4, are called the $qP1$, $qP2$, $qS1$ and $qS2$ waves, respectively (Paper I). The coefficients c_j of biquadratic equation are as follows (repeated index implies summation).

$$\begin{aligned} c_1 &= P_{ii}/g_o + 1;\\ c_2 &= (P_{ii} - Q_{ii})/g_o - T_1/g_o^2;\\ c_3 &= [\det(P_{ij})/g_o + T_2 - T_1]/g_o^2;\\ c_4 &= [\det(P_{ij}) - T_3]/g_o^3; \end{aligned} \quad \text{(A8)}$$

where,

$$\begin{aligned} T_1 &= P_{12}^2 + P_{13}^2 + P_{23}^2 - P_{11}P_{22} - P_{11}P_{33} - P_{22}P_{33},\\ T_2 &= 2(P_{12}Q_{12} + P_{13}Q_{13} + P_{23}Q_{23}) - Q_{11}(P_{22} + P_{33}) - Q_{22}(P_{11} + P_{33}) - Q_{33}(P_{11} + P_{22}),\\ T_3 &= Q_{11}\left(P_{22}P_{33} - P_{23}^2\right) + Q_{22}\left(P_{11}P_{33} - P_{13}^2\right) + Q_{33}\left(P_{11}P_{22} - P_{12}^2\right)\\ &\quad + 2Q_{12}(P_{13}P_{23} - P_{12}P_{33}) + 2Q_{13}(P_{12}P_{23} - P_{13}P_{22}) + 2Q_{23}(P_{12}P_{13} - P_{11}P_{23}). \end{aligned} \quad \text{(A9)}$$

The roots of the biquadratic equation are written as

$$\begin{aligned} h_1 &= .5(-G - L + \Delta_1);\\ h_2 &= .5(-G - L - \Delta_1);\\ h_3 &= .5(-G + L + \Delta_2);\\ h_4 &= .5(-G + L - \Delta_2); \end{aligned} \quad \text{(A10)}$$

where,

$$\Delta_1 = \sqrt{(G + L)^2 - 4(H + M)}; \quad \Delta_2 = \sqrt{(G - L)^2 - 4(H - M)};$$

$$G = -.5c_1; \quad H = \sqrt{(c_2^2/9 - c_1c_3/3 + 4c_4/3)}\cos\psi + c_2/6;$$

$$M = \sqrt{H^2 - c_4}; \quad L = (.5c_3 + GH)/M; \quad (L = \sqrt{G^2 - c_2 + 2H};\ \text{if } M = 0);$$

$$\psi = \tfrac{1}{3}\tan^{-1}(\Delta/\Gamma); \quad \Delta = \sqrt{-\Gamma^2 + (c_2^2/9 - c_1c_3/3 + 4c_4/3)^3};$$

$$\Gamma = .5c[c_3^2 + 2c_2^3/27 - c_2(c_1c_3 - 4c_4)/3 - c_4(4c_2 - c_1^2)]$$

A3 Polarizations

The polarizations (S_1, S_2, S_3) of solid particles are obtained as

$$\frac{S_1}{\Gamma_1} = \frac{S_2}{\Gamma_2} = \frac{S_3}{\Gamma_3}, \quad \text{(A11)}$$

for three sets of $(\Gamma_1, \Gamma_2, \Gamma_3)$. These sets of values of Γ_i, $(i = 1, 2, 3)$, are

(i) $\Gamma_1 = \Omega_2, \quad \Gamma_2 = \Omega_1, \quad \Gamma_3 = \Omega_{12};$
(ii) $\Gamma_1 = \Omega_3, \quad \Gamma_2 = \Omega_{13}, \quad \Gamma_3 = \Omega_1;$
(iii) $\Gamma_1 = \Omega_{23}, \quad \Gamma_2 = \Omega_3, \quad \Gamma_3 = \Omega_2;$ (A12)

where,

$$\begin{aligned} \Omega_1 &= P_{23}g_oh^2 + [P_{12}P_{13} - P_{11}P_{23} + g_o(Q_{23} - P_{23})]\,h\\ &\quad + P_{12}Q_{13} - P_{11}Q_{23} + P_{13}Q_{12} - P_{23}Q_{11} + P_{11}P_{23} - P_{12}P_{13},\\ \Omega_{23} &= g_o^2h^3 - g_o(P_{22} + P_{33} + g_o)h^2 + \left[P_{22}P_{33} - P_{23}^2 + g_o(P_{22} + P_{33} - Q_{22} - Q_{33})\right]h\\ &\quad + P_{22}Q_{33} + P_{33}Q_{22} - 2P_{23}Q_{23} + P_{23}^2 - P_{22}P_{33}. \end{aligned} \quad \text{(A13)}$$

The expression for $\Omega_2(\Omega_3)$ can be obtained from that of Ω_1 by interchanging the indices 1 and 2 (3). Similarly, the expression for $\Omega_{12}(\Omega_{13})$ can be obtained from that of Ω_{23} by replacing the index 3 (2) with 1. Use of all the three expressions given above in obtaining polarizations are discussed in Paper I. The polarization of the motion of fluid particles are defined by the relation $F_i = G_{ij}S_j$, $(i = 1, 2, 3)$, where the elements of matrix G are given by

$$G_{ij} = -r_{12}\delta_{ij} + \frac{1}{h-1}\left(\frac{1}{R}m_{ij} - r_{12}\delta_{ij}\right)n_i n_j. \tag{A14}$$