Geophys. J. Int. (2004) **158,** 225–238 doi: 10.1111/j.1365-246X.2004.02283.x

Born integral, stationary phase and linearized reflection coefficients in weak anisotropic media

Ranjit K. Shaw* and Mrinal K. Sen

Institute for Geophysics, John A. and Catherine G. Jackson School of Geosciences, 4412 Spicewood Springs Road, Bldg. 600, Austin, TX 78759, *USA.*
E-mail: mrinal@ig.utexas.edu

Accepted 2004 March 4. Received 2004 March 1; in original form 2003 April 14

SUMMARY

We show that the linearized reflection coefficients for arbitrary anisotropic media embedded in an isotropic background can be derived directly from a Born formalism. Due to rapidly varying phases of the scattered waves from first-order perturbations in density and elastic parameters, the major contributions to the observed wavefield for any source–receiver pair far from the volume of scatterers arise from the stationary points of a scattering integral, called the Born integral. For simple interface models, such integrals can be evaluated analytically using the method of stationary phase. The resulting scattering function relates linearly to the approximate (linearized) reflection coefficient through a scaling factor determined by the angle of incidence and the properties of the background medium. We consider a homogeneous isotropic background to express the approximate reflection coefficients as a sum of an isotropic and an anisotropic reflection coefficient. The isotropic coefficient is a weighted sum of density and isotropic perturbations about the background, whereas the anisotropic coefficient is a weighted sum of anisotropic perturbations where the weights depend on the angles of incidence, the properties of the background medium as well as the azimuth of the plane of reflection with respect to some symmetry plane of the weakly anisotropic medium.

We derived expressions for approximate *PP* and *PS* reflection coefficients of a weakly isotropic medium, a weakly orthorhombic medium and a weak but arbitrarily anisotropic medium underlying an isotropic medium. Our expressions for *PP* reflection coefficients are exactly the same as those obtained from first-order perturbation theory which were previously derived by linearization of the exact reflection coefficients. For converted *PS* waves, our expressions are valid only for small angles of incidence, but have much simpler forms than those obtained by linearization of exact reflection coefficients.

We also derive the approximate *PP* reflection coefficient of a transversely isotropic medium with a tilted axis of symmetry in an explicit form and investigate the effects of dip of the symmetry axis on these reflection coefficients. Numerical results demonstrate that neglecting the dip of a moderately dipping (30°–60°) symmetry axis of a transversely isotropic medium yields significant errors in determining the weakly anisotropic parameters through an analysis of variations of amplitude with azimuth.

Key words: AVOA, fractured reservoir, TTI, weak anisotropy.

1 INTRODUCTION

In seismic exploration for hydrocarbons, amplitude variations with offsets (AVO)—also called amplitude variations with angles (AVA)—contain useful information on the physical properties of potential reservoirs. Conventional AVO analysis exploits the dependence of the reflection coefficient of an interface separating two isotropic media on the *P*- and *S*-wave velocities and densities of individual media and on the incidence angle to compute the physical parameter contrasts from knowledge of a background medium characterizing the 'average' behaviour of the two media. In many real situations, the assumptions of isotropy of the media are justified. However, the presence of cracks,

*On leave from the Department of Applied Geophysics, Indian School of Mines, Dhanbad *826004,* India. E-mail: ranjitshaw@perl.ism.ac.in

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fluids, thin layers of shale, etc. often cause an otherwise isotropic reservoir to exhibit anisotropy in seismic wave propagation. The last two decades have witnessed tremendous interest among practicing geophysicists in investigating different aspects of seismic wave propagation through anisotropic media. The most common anisotropic medium to be investigated was transversely isotropic with a vertical axis of symmetry (VTI). Since in many real situations, the deviation from isotropy is only of first order, 'weak anisotropy' has emerged as a practical model for many investigations. However, in a transversely isotropic medium with a horizontal axis of symmetry (HTI), e.g. a fractured carbonate reservoir, the amplitudes of the reflected waves vary with the azimuth of the seismic line in addition to the variations with offsets. Geometrical properties of the fractures, namely the orientation, the aperture and the fracture density, as well as the physical properties of the crack infill and the background rock matrix determine the elastic coefficients. Thus, amplitude variations with azimuth (AVAZ) provide useful information to characterize the fracture properties provided that adequate azimuthal coverage is available. Seismic data acquired in a well-designed 3-D survey programme contain sufficient azimuthal coverage to perform AVAZ analysis. Furthermore, widespread availability of three-dimensional, three-component (3D-3C) data and the increasing number of shear wave exploration programmes have widened the scope for investigation of various aspects of wave propagation in anisotropic media beyond transverse isotropy with vertical or horizontal axes of symmetry.

The unusually large number of publications on anisotropic seismic wave propagation in the last decade address two different issues, namely (1) propagation characteristics, e.g. phase and normal moveout (NMO) velocities, polarization etc. (Sayers 1994; Tsvankin 1996, 1997a,b; Pšenčík & Gajewski 1998) and (2) linearization of the reflection/transmission (R/T) coefficients. The key to the application of AVO/AVAZ analyses is the knowledge of R/T coefficients of a plane wave impinging on a horizontal interface separating two media of varying density and elastic stiffness. For isotropic media with small contrasts (weak elastic), linear versions of the exact reflection coefficients (Aki & Richards 2002) serve the purpose of AVO/AVAZ analyses at small angles of incidence. In the past, the seismic industry has used a large number of such approximate relations for AVO analysis (Bortfeld 1961; Richards & Frasier 1976; Ostrander 1984; Shuey 1985). Such relations have their own domains of application. However, such simple linearizations of the R/T coefficients do not hold good for anisotropic media, even for small angles of incidence.

Another important development of the last quarter of the 20th century was the investigation into inverse scattering problems for both acoustic and elastic fields. The potential of Born/Rytov approximations in modelling migration/inversion of backward/forward scattered waves was well recognized (Stolt & Weglein 1985; Mora 1987; Rajan & Frisk 1989; Beydoun & Mendes 1989; Eaton & Stewart 1994). Clayton & Stolt (1981), Keys & Weglein (1983), Bleistein & Gray (1985), Cohen *et al.* (1986) and Bleistein (1987) used Born approximation to invert acoustic reflection data. Hudson & Heritage (1981) and Wu & Aki (1985) investigated the scattering characteristics of elastic waves scattered from an arbitrary elastic heterogeneity. Coates & Chapman (1990) showed that the Born formalism contains geometrical ray effects, namely traveltime perturbation, ray bending and focusing etc. provided that these effects are small. The work of Beylkin & Burridge (1990) on linearized inverse scattering provided a significant insight into the problem. They observed that for *PP* scattering the function characterizing the spatial distribution of scattered wave amplitudes, called the scattering function, relates to the linearized reflection coefficient of the interface through a scaling factor depending on the incidence angle. Coates & Chapman (1991) and Chapman & Coates (1994) generalized the concept of Born scattering for isotropic and anisotropic media respectively. Nicoletis *et al.* (1998) related the *PS* scattering function to the corresponding linearized reflection coefficients. Our work draws upon these observations in deriving the reflection coefficient of a weakly elastic, weakly anisotropic medium through the investigation of the scattered waves about an isotropic background medium. We further show that the scaling factors relating the scattering functions and the linearized reflection coefficients are the contributions to the Born (scattering) integral from an endpoint of the phase of the scattered waves.

An analytical expression for R/T coefficients for an interface separating two arbitrarily anisotropic media is far more difficult to derive due to the complexity of wave propagation in such media. Fryer & Frazer (1984) used stress-displacement eigenvector matrices to compute numerically the R/T coefficients over a solid–solid interface for general anisotropy. Mallick & Frazer (1991) obtained the reflection coefficients for a solid–fluid/fluid–solid interface where the solid interface exhibits azimuthal anisotropy. Their important observation that the reflection amplitudes over transversely isotropic media with a horizontal axis of symmetry vary as the square of the cosine of the azimuth angle becomes obvious from the theoretical investigations of Ruger (1997, 1998). The work of Banik (1987) contained a useful concept to characterize anisotropic media which commonly occur in seismic applications. The classic paper of Thomsen (1986) introduced the model of 'weak anisotropy' (WA) which facilitated the derivation of approximate R/T coefficients for interfaces separating different types of WA media. The most commonly used approach to derive the R/T coefficients for WA media is based on first-order perturbation of the exact R/T coefficients (Jech & Pšenčík 1989). Ruger (1998) derived the reflection coefficients for *PP* waves for interfaces separating two VTI, HTI or orthorhombic media. Pšenčík & Vavryčuk (1998), Vavryčuk & Pšenčík (1998) and Vavryčuk (1999) derived weak elastic *PP* R/T coefficients for an interface separating two weak but arbitrary anisotropic media. Zillmer *et al.* (1997,1998) obtained these coefficients for pure mode *PP* and *SS* waves. Jilek (2002) derived the reflection coefficients for converted *PS* waves for such interfaces. The approximate expressions for the converted *PS* wave reflection coefficient for arbitrarily anisotropic media are very cumbersome even under conditions of weak anisotropy. Since these derivations are based on first-order perturbation of the exact reflection coefficients, they involve a tedious sequence of algebraic exercises often performed with symbolic manipulation software only.

In this work we use the concept of a stationary phase to the first-order scattered elastic waves to arrive at the reflection coefficients for unconverted *PP* and converted *PS* (*P-SV* and *P-SH*) waves. Note that in the following we use *P-SV* and *P-SH* to mean *P-S1* and *P-S2* phases in anisotropic media. Throughout the work we use 3-D ray theoretic Green's functions. One of the major differences between our approach and others lies in the perturbed parameters. We represent the reflecting medium as a perturbation from a suitably chosen background medium, the perturbation having two components, namely weakly elastic and weakly anisotropic. Then, we investigate the condition for stationarity of the phase of the elastic waves scattered from these first-order perturbations in density and elastic stiffness. Thus, we only need to derive, at one time, the contributions from critical/endpoints of the scattering integral for the mode of reflection we are interested in as our approach is based on the contribution of the stationary phase to the specular reflection. The corresponding reflection coefficient turns out to be a weighted sum of the perturbations in density and elastic stiffness of the scattering medium about the corresponding quantities of the background. This involves much less complicated algebra and can be solved without difficulty. Moreover, the number of WA parameters governing the reflection coefficients and their sensitivities becomes obvious from the non-vanishing terms of the weighting matrix.

2 MATHEMATICAL BACKGROUND

The theory of singly scattered elastic wavefield is available in many textbooks, e.g. Ishimaru (1997), Červený (2001) etc. For the sake of brevity, we discuss the key concepts only. To derive expressions for linearized reflection coefficients of an interface separating two weakly elastic, weakly anisotropic media we start with the general case when the reflecting layer elastic parameters c_{ijkl} and density ρ can be appropriately represented by the perturbations in corresponding quantities in a background or a reference medium. Thus,

$$c_{ijkl} = c^0_{ijkl} + \Delta c_{ijkl} \text{ and } \rho = \rho^0 + \Delta\rho, \tag{1}$$

where the superscript 0 indicates the reference medium and the symbol Δ represents small perturbations, namely $|\Delta c_{ijkl}/c_{ijkl}| \ll 1$ and $|\Delta\rho/\rho| \ll 1$.

The nth component of the time-harmonic scattered wavefield at the receiver $\mathbf{x}_\mathrm{r}$ from a volume of scatterers V embedded in a background medium characterized by eq. (1) is given by

$$\Delta u_n(x_\mathrm{r}, \omega) = \int_V d\mathbf{x}' \left(\omega^2 \Delta\rho(\mathbf{x}') u^0_i(\mathbf{x}', \omega) G^0_{ni}(\mathbf{x}', \omega; \mathbf{x}_\mathrm{r}) - \Delta c_{ijkl}(\mathbf{x}') \frac{\partial u^0_k(\mathbf{x}', \omega)}{\partial x_l} \frac{\partial G^0_{ni}(\mathbf{x}', \omega; \mathbf{x}_\mathrm{r})}{\partial x_j} \right), \tag{2}$$

where $u^0_i(\cdot)$ and $G^0_{ni}(\cdot)$ refer to elastic wavefield due to the exciting source and the ray theoretic Green's function respectively.

Eq. (2) represents the singly scattered elastic wavefield under first-order perturbation of the medium parameters and is referred to as the Born integral. Next, we evaluate the Born integral (eq. 2) to derive expressions for *P-P*, *P-SV* and *P-SH* displacement reflection coefficients for an isotropic (ISO) reflector, an orthorhombic (ORT) reflector, a transversely isotropic reflector with tilted symmetry axis (TTI) and a triclinic (TRI) or arbitrary anisotropic reflector underlying an isotropic medium under the approximations of 'weak elastic and weak anisotropic' perturbations. For each case we use 3-D asymptotic ray theoretic Green's functions to compute the stationary phase of the scattered elastic waves from the volume of scatterers distributed in a 3-D array. An obvious effect of this approximation is that the derived reflection coefficients relate to plane waves and are valid for small angles of incidence.

3 LINEARIZED *PP* REFLECTION COEFFICIENTS

We consider propagation of P waves in a homogeneous and isotropic background medium with density ρ_0, P wave velocity α_0 and S wave velocity β_0. The time-harmonic Green's function representing the displacement at an observation point (receiver) $\mathbf{x}_\mathrm{r}$ due to a point source at $\mathbf{x}_\mathrm{s}$ has a simple form (Červený 2001) given by

$$G^P_{ni}(\mathbf{x}_\mathrm{r}, \omega; \mathbf{x}_\mathrm{s}) = \frac{N_n N_i}{4\pi\rho_0\alpha_0^2} \frac{1}{r} \mathrm{e}^{i\omega r/\alpha_0}, \tag{3}$$

where r is the source–receiver distance and N_n and N_i represent the source and receiver directions respectively.

Further, if the elastic field at a point $\mathbf{x}'$ is generated by a point source at $\mathbf{x}_\mathrm{s}$, we can write

$$u_{mi}(\mathbf{x}', \omega) = G_{mi}(\mathbf{x}', \omega; x_\mathrm{s}). \tag{4}$$

Now we substitute (3) and (4) into (2) to obtain an expression for the scattered P wave displacement as

$$\Delta u_{mn}(\mathbf{x}_\mathrm{r}, \omega; \mathbf{x}_\mathrm{s}) = N_m M_n \omega^2 \int_V d\mathbf{r} [(\Delta\rho\delta_{ik} + \Delta c_{ijkl} p'_j p_l) g'_i g_k] A(\mathbf{r}) \mathrm{e}^{i\omega\varphi(r)}, \tag{5}$$

where N_m and M_n are the projections of the source and the receiver directions along the incident and the scattered rays respectively with the product $N_m M_n$ representing the coupling between the source and the receiver radiation patterns. The $\mathbf{p}$ and $\mathbf{g}$ vectors represent the slowness and the polarization respectively. Primed quantities represent the scattered ray. The term $\varphi(\mathbf{r})$ denotes the accumulated phase of the ray originating at $\mathbf{x}_\mathrm{s}$ and detected at $\mathbf{x}_\mathrm{r}$ after getting scattered at $\mathbf{x}'$. $A(\mathbf{r})$ stands for a scalar amplitude given by

$$A(\mathbf{r}) = \frac{1}{\left(4\pi\rho_0\alpha_0^2\right)^2} \frac{1}{r^2}. \tag{6}$$

The term inside the square brackets in eq. (5) determines the scattering pattern due to perturbations in the physical properties of the medium and is called the 'scattering function' (Eaton & Stewart 1994; Burridge *et al.* 1998).

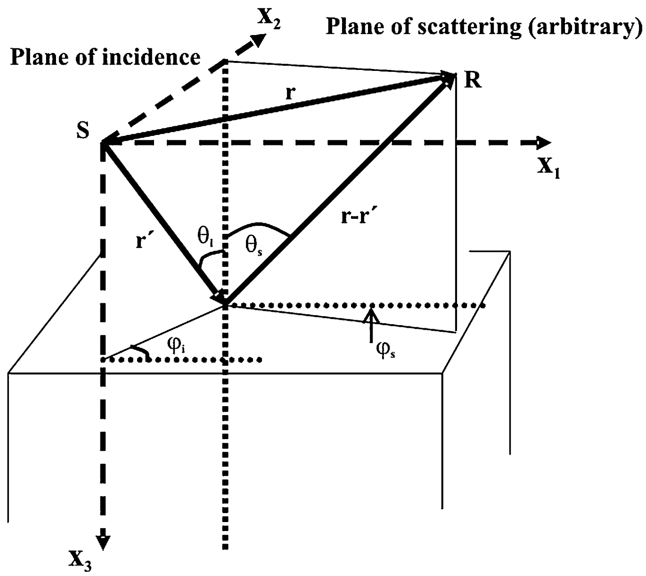

Figure 1. Geometry of elastic scattering. The origin of the coordinate system is at the source. θ_{i} and ϕ_{i} are the polar and azimuth angles respectively of the incident ray from S. θ_{s} and ϕ_{s} are the corresponding angles for the scattered ray received at R.

A close look at eqs (5) and (6) shows that at distances far from the volume of scatterers, the amplitudes of the scattered waves vary relatively slowly compared with the large but rapid oscillations of the corresponding phases. This facilitates the use of the method of stationary phase (see Appendix A and Fig. 2) to evaluate the Born integral (eq. 5). To meet this objective, we rewrite eq. (5) as

$$\Delta u_{mn}(\mathbf{x}_{\mathrm{r}},\omega;\mathbf{x}_{\mathrm{s}}) = N_m M_n \omega^2 \int_Z dx_3' \left(\int\int_S dx_1'\, dx_2' [(\Delta\rho\delta_{ik} + \Delta c_{ijkl} p_j' p_l) g_i' g_k] A(\mathbf{r}) \mathrm{e}^{i\omega\phi(\mathbf{r})} \right)$$
$$= N_m M_n \omega^2 \int_Z I dx_3', \tag{7}$$

where

$$I = \int\int_S dx_1' dx_2' \left[\left(\Delta\rho\delta_{ik} + \Delta c_{ijkl} p_j' p_l\right) g_i' g_k \right] A(\mathbf{r}) \mathrm{e}^{i\omega\varphi(\mathbf{r})}. \tag{8}$$

Referring to a rectangular Cartesian coordinate system (Fig. 1) with its origin at the source S and radius vectors of the receiver R and an arbitrary scatterer Sc denoted by $\mathbf{r}$ and $\mathbf{r}'$ respectively, we write the phase term for a P wave travelling from S to R scattered at Sc, as

$$\varphi(\mathbf{r}) = \frac{1}{\alpha_0}(|\mathbf{r}'| + |\mathbf{r} - \mathbf{r}'|). \tag{9}$$

The unit vectors along $\mathbf{r}'$ and $\mathbf{r} - \mathbf{r}'$ are

$$\hat{\mathbf{r}}_1 = \sin\theta_{\mathrm{i}} \cos\phi_{\mathrm{i}}\, \hat{\mathbf{i}} + \sin\theta_{\mathrm{i}} \sin\phi_{\mathrm{i}}\, \hat{\mathbf{j}} + \cos\theta_{\mathrm{i}}\, \hat{\mathbf{k}}$$

and

$$\hat{\mathbf{r}}_2' = \sin\theta_{\mathrm{s}} \cos\phi_{\mathrm{s}}\, \hat{\mathbf{i}} + \sin\theta_{\mathrm{s}} \sin\phi_{\mathrm{s}}\, \hat{\mathbf{j}} + \cos\theta_{\mathrm{s}}\, \hat{\mathbf{k}}$$

respectively where the angles θ and ϕ represent polar and azimuth angles respectively with the suffix i indicating incident wave and the suffix s denoting the scattered wave. The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ refer to the X_1, X_2 and X_3 axes of the coordinate system respectively.
The condition for the stationary phase in the horizontal plane ($x_1 - x_2$) is given by $\nabla\varphi(\mathbf{r}) = 0$ which leads to

$$-\sin\theta_{\mathrm{i}} \cos\phi_{\mathrm{i}} + \sin\theta_{\mathrm{s}} \cos\phi_{\mathrm{s}} = 0$$

$$-\sin\theta_{\mathrm{i}} \sin\phi_{\mathrm{i}} + \sin\theta_{\mathrm{s}} \sin\phi_{\mathrm{s}} = 0$$

implying

$$\theta_{\mathrm{s}} = \theta_{\mathrm{i}} \quad \text{and} \quad \phi_{\mathrm{s}} = \pi + \phi_{\mathrm{i}}, \tag{10}$$

i.e. the source, the receiver and the scatterer lie in a vertical plane with the incident angle equalling the scattering angle for specular reflection which is Snell's law of reflection from a horizontal plane in geometrical optics.

The surface integral (8) is approximated as (Appendix A)

$$I = \frac{i\pi\alpha_0 r}{\omega\cos i}\, [\Delta\rho\delta_{ik} + \Delta c_{ijkl} p_j' p_l] g_i' g_k A(\mathbf{r}) \mathrm{e}^{i\omega\varphi(\mathbf{r})}\Big|_{\mathbf{r}=\mathbf{r}_0'}. \tag{11}$$

Here, $\mathbf{r} = \mathbf{r}'_0(x_1, x_2)$ represents the stationary point of the integral (8) where the gradient of the phase $\varphi(r)$ vanishes.

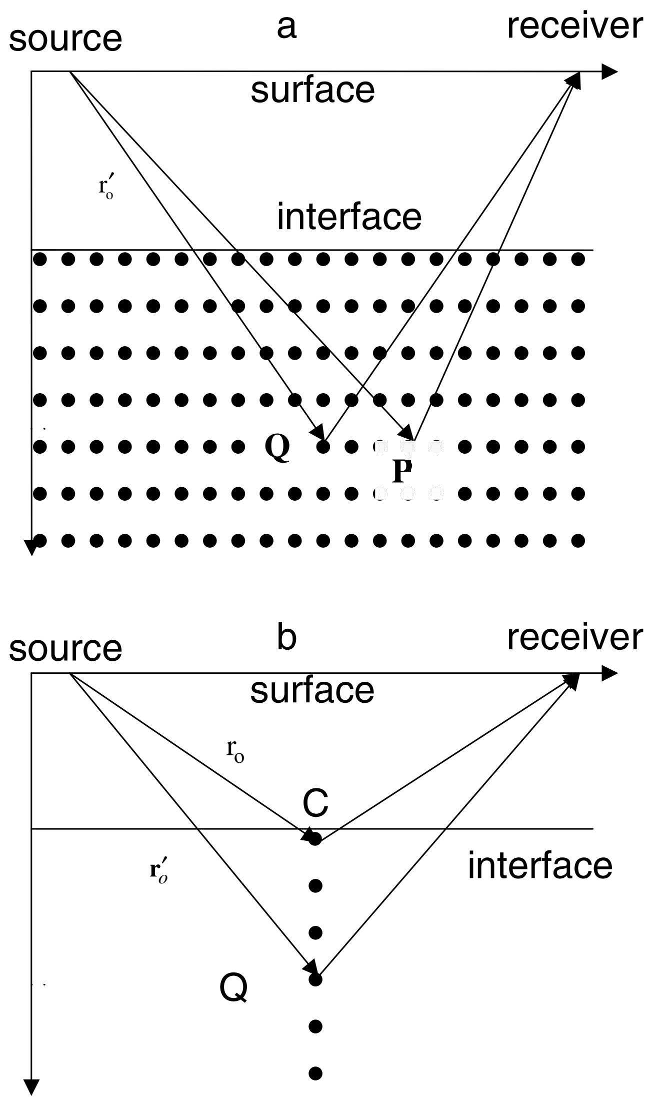

Figure 2. (a) Stationary point of a Born integral. For a source–receiver pair, the contribution to the overall scattered amplitude from all the scatterers at a specific depth level except the one corresponding to the stationary phase point (Q) contribute insignificantly to the Born integral (b) The endpoint (C) has the maximum contribution to the integral obtained after consideration of the stationary phase.

We substitute the result (11) into (7) to find the far-field asymptotic solution as (Appendix A)

$$\Delta u_{mn}(\mathbf{x}_r, \omega; \mathbf{x}_s) = -N_m M_n \frac{1}{4\pi\rho_0\alpha_0^2 r_0}\left(\frac{1}{4\rho_0\cos^2 i}S(\mathbf{r}_0)\right)\mathrm{e}^{i\omega\varphi(\mathbf{r}_0)}, \tag{12}$$

where

$$S(\mathbf{r}_0) = [\Delta\rho\delta_{ik} + \Delta c_{ijkl}p'_j p_l]g'_i g_k|_{\mathbf{r}=\mathbf{r}'_0} \tag{13}$$

is the value of the scattering function corresponding to the endpoint $\mathbf{r} = \mathbf{r}_0$ of (7). Fig. 2 explains the procedure for evaluating eq. (7). Rearranging the terms of eq. (12) we have

$$\Delta u_{mn}(\mathbf{x}_r, \omega; \mathbf{x}_s) = -G_{mn}(\mathbf{x}_r, \omega; \mathbf{x}_s)R(i). \tag{14}$$

Comparing eq. (14) with the ray-theoretical Green's function for a reflected *PP* wave (Červený 2001), we identify $R(i)$ to be the approximate plane wave reflection coefficient for an angle of incidence i (Aki & Richards 2002). Thus, the reflection coefficient and the scattering functions are related through the following simple equation

$$R(i) = \frac{1}{4\rho_0\cos^2 i}S(\mathbf{r}_0) \tag{15}$$

We emphasize that more than a decade ago Beylkin & Burridge (1990) observed that the linearized *PP* reflection coefficient is a scaled version of the scattering function. Eq. (15), indeed, validates that statement. However, we show later that the scaling factor is different for different modes of reflection.

Throughout this work, we consider both the background as well as the reflecting media to be homogeneous. This allows us to write eq. (13) as

$$S(\mathbf{r}_0) = \Delta\rho\xi + \Delta c_{mn}\eta_{mn}, \tag{16}$$

where

$$\xi = [g_i\, g_i']_{\mathbf{r}=\mathbf{r}_0} \tag{17}$$

and

$$\eta_{mn} = [g_i' p_j' g_k p_l]_{\mathbf{r}=\mathbf{r}_0}. \tag{18}$$

The suffixes m and n refer to Voigt's concise notation, m takes values over i and j whereas n takes values over k and l with $i, j, k, l = 1$, 2, 3.The summation convention over repeated indices applies to eq. (16).

4 RESULTS

Before proceeding to derive any result, we address the crucial issue of selecting the background medium. Mensch & Rasolofosaon (1997) and Pšenčík & Gajewski (1998) investigated this issue in detail and concluded that one can use the prevailing direction of P wave propagation in the anisotropic medium for the experiment under consideration as a guide to select the P wave velocity of the background medium. To illustrate the point further, the best choice for the background P wave velocity for reflection seismics is $\rho_0\alpha_0^2 = c_{33}$ and for cross-well experiments is $\rho_0\alpha_0^2 = c_{11}$ or c_{22} representing predominantly vertical and horizontal propagation respectively. The best choice of S wave velocity for lower-order symmetry, e.g. an HTI medium, is the vertical velocity with polarization in the plane of isotropy. Pšenčík & Martin (2001) investigated the effects of background velocities on the accuracy and approximations of the reflection coefficients and concluded that for small angles of incidence the choice of vertical velocity is the most appropriate. Further, it is easy to appreciate that for large angles, first-order R/T coefficients usually lose their accuracy due to neglected higher-order terms. Jilek (2002) argued in favour of selecting an isotropic background for complicated models, such as those with arbitrary orientation of symmetry axes/symmetry planes for TI/orthorhombic media around the interface. However, we found that such a preference for the background medium in our approach only has an effect on the required algebra and the structure of the isotropic and anisotropic part of the reflection coefficients. Keeping reflection seismics in mind, we choose $\rho_0\alpha_0^2 = c_{33}^0$ and $\rho_0\beta_0^2 = c_{55}^0$ unless otherwise stated.

4.1 An isotropic medium

If both the background and the reflecting media are isotropic, then the perturbation in the elastic stiffness tensor is

$$\begin{bmatrix} \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} - 2\Delta c_{55} & & & \\ \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & & & \\ \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} & & & \\ & & & \Delta c_{55} & & \\ & & & & \Delta c_{55} & \\ & & & & & \Delta c_{55} \end{bmatrix} \tag{19}$$

where Δc_{33} and Δc_{55} are the perturbations in the corresponding elastic parameters, namely the Lame parameters in this case.

We substitute (19) into (16) to obtain

$$S(\mathbf{r}_0) = \Delta\rho\cos 2i + \frac{1}{\alpha_0^2}\left(\Delta c_{33} - \Delta c_{55}\sin^2 2i\right)$$

which corresponds, from eq. (15), to a linearized reflection coefficient given by

$$R_{PP}^{\mathrm{ISO}}(i) = \frac{1}{2}\frac{\Delta Z}{Z_0} + \frac{1}{2}\left[\frac{\Delta\alpha}{\alpha_0} - 4\left(\frac{\beta_0^2}{\alpha_0^2}\right)\frac{\Delta G}{G_0}\right]\sin^2 i + \frac{1}{2}\frac{\Delta\alpha}{\alpha_0}\sin^2 i\tan^2 i, \tag{20}$$

where Z_0 and G_0 represent the P-wave impedance and shear modulus in the background medium and ΔZ and ΔG represent the corresponding perturbations. If we consider a background medium with density and elastic parameters as the average of the corresponding values in the two media, eq. (20) is exactly the same as the linearized PP reflection coefficient for isotropic media given in Aki & Richards (2002).

4.2 An orthorhombic medium

Ruger (1998) mentioned that Corrigan (1990) first derived an expression for the reflection coefficient for an orthorhombic medium using the Born approximation. However, to the best of our knowledge this work was never published. Ruger (1998) compared his expressions with those of Corrigan's. We give here a full treatment of the subject.

We parametrize the elastic stiffness of the reflecting weakly orthorhombic medium as a perturbation over a homogeneous isotropic background with elastic constants c_{33}^0 and c_{55}^0 as follows:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{55} & \\ & & & & & c_{66} \end{bmatrix} = \begin{bmatrix} c_{33}^0 & c_{33}^0 - 2c_{55}^0 & c_{33}^0 - 2c_{55}^0 & & & \\ c_{33}^0 - 2c_{55}^0 & c_{33}^0 & c_{33}^0 - 2c_{55}^0 & & & \\ c_{33}^0 - 2c_{55}^0 & c_{33}^0 - 2c_{55}^0 & c_{33}^0 & & & \\ & & & c_{55}^0 & & \\ & & & & c_{55}^0 & \\ & & & & & c_{55}^0 \end{bmatrix}$$
$$+ \begin{bmatrix} \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} - 2\Delta c_{55} & & & \\ \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & & & \\ \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} & & & \\ & & & \Delta c_{55} & & \\ & & & & \Delta c_{55} & \\ & & & & & \Delta c_{55} \end{bmatrix}$$
$$+ \begin{bmatrix} c_{11} - c_{33} & c_{12} - c_{33} + 2c_{55} & c_{13} - c_{33} + 2c_{55} & & & \\ c_{12} - c_{33} + 2c_{55} & c_{22} - c_{33} & c_{23} - c_{33} + 2c_{55} & & & \\ c_{13} - c_{33} + 2c_{55} & c_{23} - c_{33} + 2c_{55} & 0 & & & \\ & & & c_{44} - c_{55} & & \\ & & & & 0 & \\ & & & & & c_{66} - c_{55} \end{bmatrix}, \tag{21}$$

where $\Delta c_{33} = c_{33} - c_{33}^0$ and $\Delta c_{55} = c_{55} - c_{55}^0$. The stiffness coefficients in the square brackets on the right-hand side represent isotropic background, weakly isotropic perturbations and weakly anisotropic perturbations respectively.

Substituting expression (21) in (15), we obtain the linearized *PP* reflection coefficient over a weakly orthorhombic medium as

$$R_{PP}^{\mathrm{ORT}}(i,\phi) = {}^{\mathrm{iso}}R_{PP}^{\mathrm{ORT}}(i) + {}^{\mathrm{ani}}R_{PP}^{\mathrm{ORT}}(i,\phi), \tag{22}$$

with

$$\begin{aligned} {}^{\mathrm{ani}}R_{PP}^{\mathrm{ORT}}(i,\phi) = {} & \frac{1}{2}\left[\delta_1 \cos^2\phi + \left(\delta_2 - 8\frac{\beta_0^2}{\alpha_0^2}\gamma\right)\sin^2\phi\right]\sin^2 i \\ & + \frac{1}{2}\left[\varepsilon_1 \cos^4\phi + \varepsilon_2 \sin^4\phi + \delta_3 \sin^2\phi\cos^2\phi\right]\sin^2 i \tan^2 i \end{aligned} \tag{23}$$

where

$$\varepsilon_1 = \frac{c_{11} - c_{33}}{2c_{33}}, \qquad \varepsilon_2 = \frac{c_{22} - c_{33}}{2c_{33}}, \qquad \gamma = \frac{c_{44} - c_{55}}{2c_{55}},$$
$$\delta_1 = \frac{c_{13} - c_{33} + 2c_{55}}{c_{33}}, \; \delta_2 = \frac{c_{23} - c_{33} + 2c_{44}}{c_{33}} \quad \text{and} \quad \delta_3 = \frac{c_{12} - c_{33} + 2c_{66}}{c_{33}}. \tag{24}$$

are the weak anisotropy parameters defined by Pšenčík & Vavryčuk (1998) for orthorhombic medium. ${}^{\mathrm{iso}}R_{PP}^{\mathrm{ORT}}(i)$ in eq. (22) is exactly the same as in eq. (20).

Eq. (22) reduces to eqs (23) and (24) of Ruger's (1998) expressions for $\phi = 0^\circ$ and $\phi = 90^\circ$ respectively. Note the change in definition of the anisotropy parameters by Ruger (1998) and Pšenčík & Vavryčuk (1998). It is worth mentioning that our results are more general than those of Corrigan (cf. eq. 26 in Ruger 1998) and are not restricted to very small angles of incidence.

4.3 A TTI medium

We obtain the elastic stiffness tensor of a transversely isotropic medium with a symmetry axis inclined at an angle θ in $x_1 - x_3$ plane through the Bond transformation (Auld 1990) as

$$C' = \begin{bmatrix} c'_{11} & c'_{12} & c'_{13} & 0 & c'_{15} & 0 \\ & c'_{22} & c'_{23} & 0 & c'_{25} & 0 \\ & & c'_{33} & 0 & c'_{35} & 0 \\ & & & c'_{44} & 0 & c'_{46} \\ & & & & c'_{55} & 0 \\ & & & & & c'_{66} \end{bmatrix}, \tag{25}$$

where the primed quantities denote elastic parameters in the rotated coordinate system.

In this case the algebra gets simplified if we consider the background isotropic medium with stiffness c'_{33} and c'_{44}. The final expression for linear *PP* reflection coefficient has the following form:

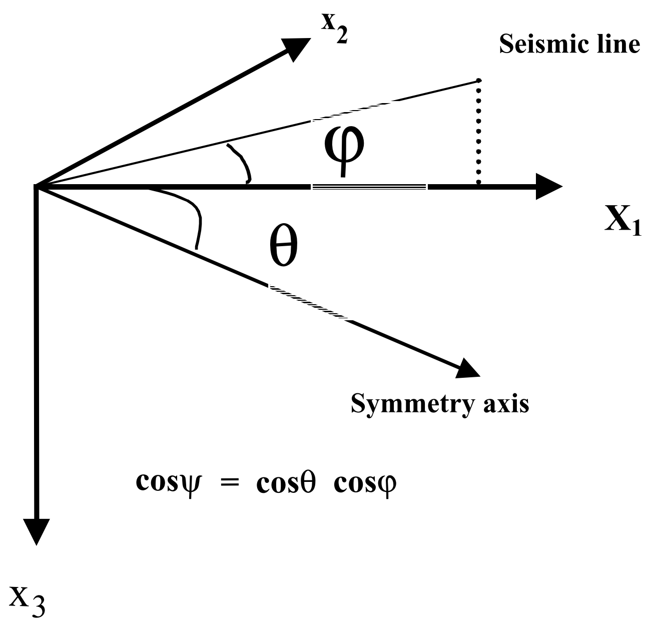

Figure 3. Tilted transverse isotropy (TTI): the symmetry axis of the medium is tilted by an angle θ with the x_1 axis in the $x_1 - x_3$ plane.

$$R_{PP}^{\mathrm{TTI}}(i,\phi) = {}^{\mathrm{iso}}R_{PP}^{\mathrm{TTI}}(i) + {}^{\mathrm{ani}}R_{PP}^{\mathrm{TTI}}(i,\phi), \tag{26}$$

where

$${}^{\mathrm{iso}}R_{PP}^{\mathrm{TTI}}(i) = \frac{1}{2}\frac{\Delta Z'}{Z'_0} + \frac{1}{2}\left[\frac{\Delta\alpha'}{\alpha'_0} - 4\left(\frac{\beta_0'^2}{\alpha_0'^2}\right)\frac{\Delta\tilde{G}'}{\tilde{G}'_0}\right]\sin^2 i + \frac{1}{2}\frac{\Delta\alpha'}{\alpha'_0}\sin^2 i\tan^2 i, \tag{27}$$

and

$$\begin{aligned}{}^{\mathrm{ani}}R_{PP}^{\mathrm{TTI}}(i,\phi) = \Bigg[&\left(\frac{\delta}{2}(\cos^2\psi - \sin^2\theta) - (\varepsilon-\delta)(\cos^2\psi + \sin^2\theta)\sin^2\theta + 4\frac{\beta_o^2}{\alpha_o^2}\gamma\cos^2\psi\right)\sin^2 i \\ &+\frac{1}{2}\left[\delta(\cos^2\psi - \sin^2\theta) + (\varepsilon-\delta)(\cos^4\psi - \sin^4\theta)\right]\sin^2 i\tan^2 i\Bigg].\end{aligned} \tag{28}$$

Here, $\cos\psi = \cos\theta\cos\phi$ (Fig. 3) is the cosine of the angle between the seismic line and the symmetry axis of the weak TTI medium measured in the plane containing these lines.

For $\theta = 0$, $\cos\psi = \cos\phi$ and eq. (26) reduces to the linearized reflection coefficient over a HTI medium with a symmetry axis along the x_1 direction. Similarly for $\theta = \pi/2$, $\cos\psi = 0$, eq. (26) becomes the linearized reflection coefficients for a VTI medium

$${}^{\mathrm{ani}}R_{PP}^{\mathrm{TTI}}(i) = \frac{1}{2}\left(\delta^v\sin^2 i + \varepsilon^v\sin^2 i\tan^2 i\right), \tag{29}$$

where $\delta^v = \delta - 2\varepsilon$ and $\varepsilon^v = -\varepsilon$ are the Thomsen weak anisotropy parameters for a VTI medium.

4.4 A general anisotropic (triclinic) medium

We used the same procedure as above to derive the linearized *PP* reflection coefficient over a weakly triclinic medium. The corresponding anisotropic reflection coefficient is

$$\begin{aligned}{}^{\mathrm{ani}}R_{PP}^{\mathrm{TTI}}(i,\phi) = \Bigg[&\frac{1}{2}\left(\delta_1\cos^2\phi + \delta_2\sin^2\phi - 8\frac{\beta_0^2}{\alpha_0^2}\gamma\sin^2\phi + 2(\varepsilon_{36} - 2\varepsilon_{45})\sin\phi\cos\phi\right)\sin^2 i \\ &+\frac{1}{2}\left[\varepsilon_1\cos^4\phi + \varepsilon_2\sin^4\phi + \delta_3\sin^2\phi\cos^2\phi + 2(\varepsilon_{16}\cos^2\phi + \varepsilon_{26}\sin^2\phi)\sin\phi\cos\phi\right]\sin^2 i\tan^2 i\Bigg]\end{aligned} \tag{30}$$

where the additional WA parameters are

$$\varepsilon_{16} = \frac{c_{16}}{c_{33}};\quad \varepsilon_{26} = \frac{c_{26}}{c_{33}};\quad \varepsilon_{36} = \frac{c_{36}}{c_{33}};\quad \varepsilon_{45} = \frac{c_{45}}{c_{33}}. \tag{31}$$

The isotropic part of the linear reflection coefficient is the same as eq. (20).

Eq. (30) is the same as the anisotropic part of eq. (40) of Vavryčuk & Pšenčík (1998) for a general anisotropic reflector underlying an isotropic medium. We emphasize at this stage that the *PP* reflection coefficient given in eq. (30) depends on 10 WA parameters defined by eqs (24) and (31), instead of 15 possible parameters identified by Pšenčík & Vavryčuk (1998).

Table 1. Parameters of an interface separating a HTI half-space from an isotropic overburden.

Medium	α (km s^{-1})	β (km s^{-1})	ρ (g cm^{-3})	ε	δ	γ
Isotropic	4.00	2.31	2.65	0.000	0.000	0.000
HTI	3.07	2.06	2.60	−0.191	−0.238	0.127

5 TEST EXAMPLE

Ruger (1997, 1998), Pšenčík & Vavryčuk (1998) and Vavryčuk & Pšenčík (1998) compared the approximate *PP* reflection coefficients with the exact values for different situations. As our expressions for approximate *PP* reflection coefficients are the same as those reported in the cited works, we do not repeat any further tests here. Rather, we investigate the effect of dip of the symmetry axis of a TTI medium on the approximate *PP* reflection coefficient. We consider an interface separating a HTI half-space from an isotropic overburden with density and elastic parameters described in Table 1. The numerical experiments consist of rotating the elastic coefficients of the HTI medium for 0, 30, 45, 60 and 90° and computing the *PP* reflection coefficients from eq. (26). Fig. 4 displays the results, which show that the tilt of the symmetry axis of a TTI medium has a significant effect on the *PP* reflection coefficients. Fig. 4(f) shows the error in approximate reflection coefficient if the same TTI medium with a 30° dip of symmetry axis is interpreted as a HTI medium. This shows that neglecting the dip of the symmetry axis of a TTI medium in AVAZ analysis may affect the estimated AVO gradient significantly.

6 DISCUSSONS AND CONLUSIONS

Analytical reflection coefficients in anisotropic media are available for very high orders of symmetry. However, these expressions are too complex to understand the role of various factors controlling the reflection amplitudes. Linearized approximations, on the other hand, have many practical applications in analysing AVO and AVAZ. Conventionally, these approximations are achieved under the assumption of small contrasts in density and elastic parameters across the reflecting interface. First-order perturbation theory has been extensively used to linearize the reflection coefficient over an arbitrary weakly anisotropic medium. In this work, we formulated the problem of reflection from a weakly elastic, weakly anisotropic medium as that of scattering of elastic waves about a homogeneous isotropic background and used the Born approximation to derive the approximate reflection coefficient. The approximate reflection coefficient turns out to be a scaled version of the scattering function corresponding to the stationary phase value of the Born integral. The scaling factor depends on the mode of reflection, namely pure modes or converted waves, and on the angles of incidence and reflection. We characterize a weakly elastic, weakly anisotropic medium as the superposition of an isotropic background, a weakly elastic (isotropic) perturbation and a weakly anisotropic perturbation. Thus, the approximate reflection coefficients have two components, isotropic and anisotropic. The anisotropic reflection coefficient appears as the weighted sum of the anisotropic perturbations with the weighting factors depending on the angles of incidence and reflection as well as the azimuth of the plane of reflection with respect to the symmetry axis of the reflecting medium. The structure of the weighting matrix specifies the importance and sensitivity of different anisotropic perturbations in defining the anisotropic reflection coefficient. For *PP* reflections, approximate reflection coefficients from our study match exactly with those obtained by previous workers for different kinds of weak anisotropic media. For converted waves, we obtain simpler expressions for the anisotropic reflection coefficients for small incidence angles. We also derived an expression for the approximate *PP* reflection coefficient for a transverse isotropic medium with a tilted symmetry axis and showed through numerical examples the importance of considering the tilt of the symmetry axis in AVAZ analysis.

ACKNOWLEDGMENTS

This work was partially supported by a grant from the US Department of Energy contract # DE-FC26-00BC15305. RKS thanks Professor B. B. Bhattacharya, Ex-Director, Indian School of Mines, Dhanbad for encouragement to take up this work and for granting leave of absence. Comments and suggestions from an anonymous reviewer and the associate editor improved the manuscript considerably. R. K. Shaw was partially supported by a University of Texas Institute for Geophysics post-doctoral fellowship. University of Texas Institute for Geophysics Contribution Number 1684.

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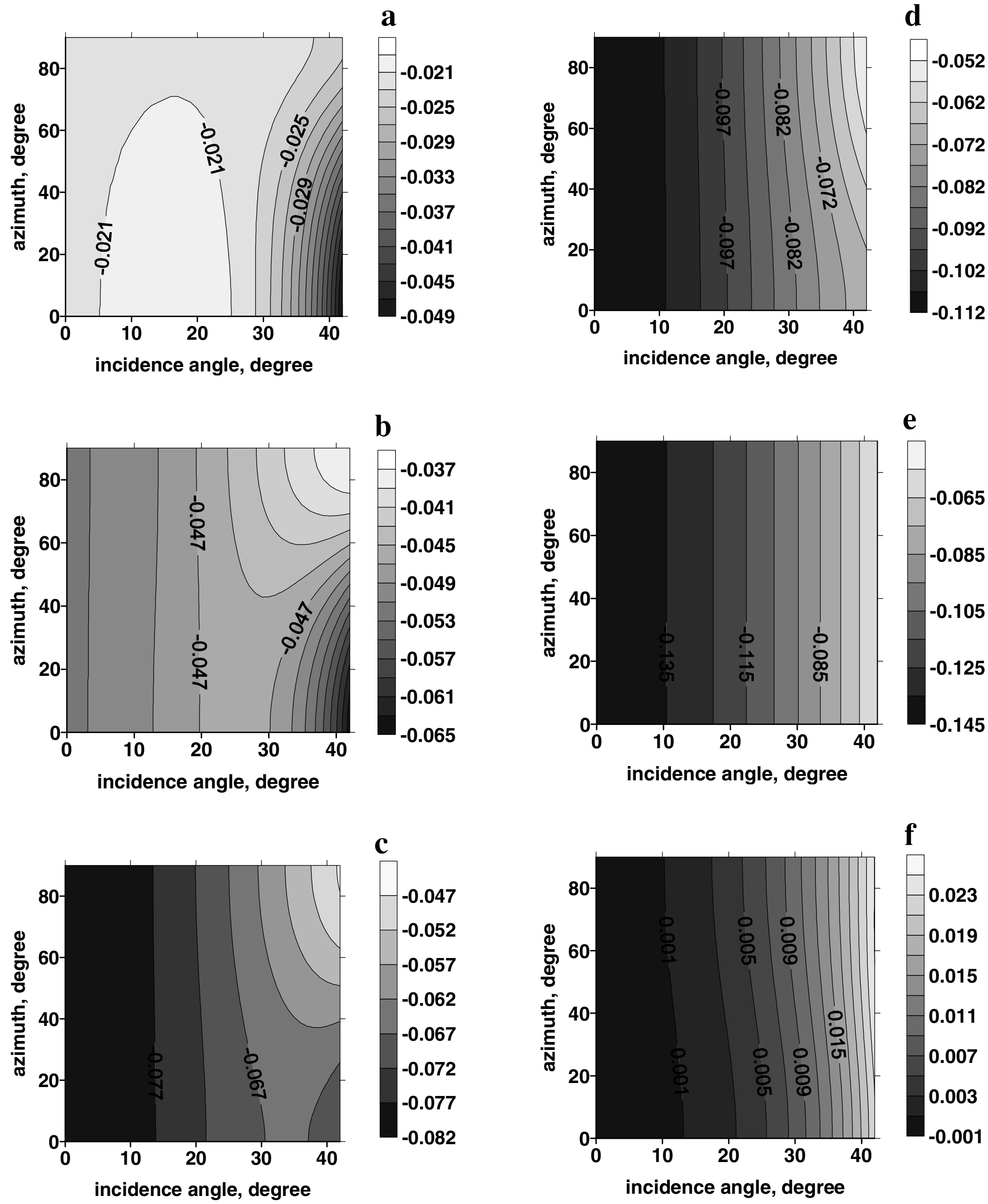

Figure 4. Effect of dip of the symmetry axis of a TTI medium on the linearized reflection coefficients: The approximate reflection coefficients of a TTI medium with dip of symmetry axis (a) 0° (HTI), (b) 30°, (c) 45°, (d) 60°, (e) 90° (VTI). (f) shows the residual error in reflection coefficient if a TTI with dip of symmetry axis 30° is modelled as an HTI medium.

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APPENDIX A: APPROXIMATE EVALUATION OF THE BORN INTEGRAL

In this appendix we evaluate the Born integral using the stationary phase method. We consider an *m*-dimensional Fourier transform of a function $f(r)$ as

$$F(\omega) = \int_{-\infty}^{\infty} f(r) \mathrm{e}^{i\omega\varphi(r)}\, dr. \tag{A1}$$

We assume that the function $f(r)$ varies relatively slowly with respect to r where as the function $\varphi(r)$ is large and rapidly varying. In the high-frequency approximation, the phase term in the integrand of (A1), namely $\mathrm{e}^{i\omega\varphi(r)}$ shows rapid oscillations in the most of the domain of integration, resulting in a contribution of practically zero to the integral. An exception to this occurs where $\varphi(r)$ exhibits an extremum, which is a stationary point. Thus, the integral (A1) can be evaluated from the sum of the contributions arising from all stationary points.

Supposing that the phase $\varphi(r)$ has a stationary point at $r = r_0$, it can be expanded in the form of a Taylor series in the vicinity of the stationary point as

$$\varphi(r) = \varphi(r_0) + \frac{1}{2}\nabla^2\, \varphi(r)|_{r_0}\, (r - r_0)^2 + \cdots. \tag{A2}$$

Since $f(r)$ varies slowly, eq. (A1) approximates as

$$F(\omega) \approx f(r_0)\mathrm{e}^{i\omega\varphi(r_0)} \int_{-\infty}^{\infty} \mathrm{e}^{i\,\frac{1}{2}\nabla^2\,\varphi(r)|_{r_0}(r-r_0)^2}\, dr. \tag{A3}$$

Eq. (A3) reduced to (Bleistein 1984)

$$F(\omega) \approx \left(\frac{2\pi}{\omega}\right)^{\frac{m}{2}} \frac{f(r_0)}{\sqrt{|\det \mathbf{A}|}} \mathrm{e}^{i\omega\varphi(r_0)+i\,\mathrm{sgn}A\pi/4} \tag{A4}$$

for det $\mathbf{A} \neq 0$.The Hessian matrix $\mathbf{A}$ is obtained as

$$A_{ij} = \left(\frac{\partial^2\varphi(r_0)}{\partial x_i \partial x_j}\right), i, j = 1, 2, \ldots m, \tag{A5}$$

with the signature of the matrix $\mathbf{A}$, sgn $\mathbf{A}$, defined as the difference of the numbers of positive and negative eigenvalues of the matrix $\mathbf{A}$. Here, r_0 represents the stationary point.

We substitute eq. (9) in eq. (A5) to obtain

$$\det \mathbf{A} = \frac{4\cos^2 i}{\alpha_0^2 r_0'^2} \tag{A6}$$

and sgn $\mathbf{A} = 2$, as the Hessian matrix has two positive eigenvalues and no negative eigenvalues.

Now, we substitute eq. (A6) in (A4) and obtain

$$F(\omega) = \frac{\pi i \alpha_0 r}{\omega \cos i}\, [\Delta\rho\delta_{ik} + \Delta c_{ijkl} p_j' p_l] g_i' g_k A(r) \mathrm{e}^{i\omega\phi(r)}\big|_{r=r_0'} \tag{A7}$$

which is eq. (11).

We further consider the asymptotic expansion of the integral of type

$$H(\omega) = \int_\lambda^\infty h(r) \mathrm{e}^{i\omega\varphi(r)}\, dr \tag{A8}$$

where λ is an endpoint (Bleistein 1984). The approximation to integral (A8) is obtained as

$$H(\omega) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(i\omega)^{n+1}} \left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^n \left(\frac{h(r)}{\varphi'(r)}\right) \mathrm{e}^{i\omega\varphi(r)}\Bigg|_{r=\lambda}. \tag{A9}$$

For a zeroth-order ray approximation, eq. (A9) reduces to

$$H(\omega) = \frac{-1}{i\omega}\frac{h(r)}{\varphi'(r)} \mathrm{e}^{i\omega\varphi(r)}\Bigg|_{r=\lambda}. \tag{A10}$$

We substitute eq. (A7) in eq. (7) and compare the results with eq. (A8). Using (A10), we finally obtain approximation for eq. (7) in the form of eq. (12).

APPENDIX B: LINEARIZED *PS* REFLECTION COEFFICIENTS

To derive the linearized reflection coefficient for a *PS* converted wave for different types of anisotropic media from the corresponding scattering function, we follow the same procedure as that outlined for the unconverted *PP* waves. The relation is expressed as

$$R_{PS} = \frac{\sin i}{2\rho_0 \cos j \sin(i+j)} S(\mathbf{r}_0). \tag{B1}$$

We tabulate here the final expressions for linearized *P-SV* and *P-SH* reflection coefficients over a weakly isotropic, a weakly orthorhombic and a weakly triclinic media.

B1 *P-SV* reflection coefficients

B1.1 Isotropic medium

$$R_{PSV}^{\mathrm{ISO}} = -\frac{\sin i}{2\cos j}\left[\frac{\Delta\rho}{\rho_0} + 2\frac{\beta_0}{\alpha_0}\frac{\Delta G}{G_0}\cos i \cos j - 2\left(\frac{\beta_0}{\alpha_0}\right)^2 \frac{\Delta G}{G_0}\sin^2 i\right] \tag{B2}$$

which readily reduces to the expression for approximate *PS* reflection coefficient of Aki & Richards (2002).

B1.2 Orthorhombic medium

The anisotropic part of *P-SV* reflection coefficient in an orthorhombic medium turns out to be

$$\begin{aligned} {}^{\mathrm{ani}}R_{PSV}^{\mathrm{ORT}} = \frac{\alpha_0}{\beta_0}\frac{\sin i \sin j}{\sin(i+j)}\Bigg(&\frac{1}{2}\left[\delta_1\cos^2\phi + \left(\delta_2 - 4\frac{\beta_0^2}{\alpha_0^2}\gamma\right)\sin^2\phi\right] - 4\frac{\beta_0^2}{\alpha_0^2}\gamma \sin i \cos i \sin^2\phi \\ &+ \left\{\left(\varepsilon_1\cos^4\phi + \varepsilon_2\sin^4\phi\right) - \left[\delta_1\cos^2\phi + \delta_2\sin^2\phi - \left(\delta_3 - 4\frac{\beta_0^2}{\alpha_0^2}\gamma\right)\sin^2\phi\cos^2\phi\right]\right\}\sin^2 i\Bigg) \end{aligned} \tag{B3}$$

and the isotropic part is given by (B2).

B1.3 General anisotropic medium

The anisotropic part of the *P-SV* reflection coefficient, for small incidence angles, is

$${}^{\mathrm{ani}}R_{PSV}^{\mathrm{TRI}} = \frac{\alpha_0}{\beta_0}\frac{1}{\sin(i+j)}\left(\sum_{k=1}^{7}\nu_k V_k\right) \tag{B4}$$

where

$$\nu_1 = \frac{1}{2}\sin i \sin j$$
$$\nu_2 = \sin^3 i \sin j$$
$$\nu_3 = -\frac{\sin^2 i \cos i \cos 2j}{\cos j}$$
$$\nu_4 = \frac{1}{2}\frac{\sin^2 i \sin(2j - i)}{\cos j}$$
$$\nu_5 = -\frac{1}{2}\frac{\sin i \cos i \sin(2j - i)}{\cos j}$$
$$\nu_6 = \frac{\sin i \sin(2j - \kappa_1)}{\cos j}$$
$$\nu_7 = \frac{\sin i \sin(2j - \kappa_2)}{\cos j}.$$

and

$$V_1 = \delta_1 \cos^2\phi + \left(\delta_2 - 8\frac{\beta_0^2}{\alpha_0^2}\gamma\right)\sin^2\phi + 2\Delta\varepsilon_{36}\sin\phi\cos\phi$$
$$V_2 = \left(\varepsilon_1\cos^4\phi + \varepsilon_2\sin^4\phi\right) - \left(\delta_1\cos^2\phi + \delta_2\sin^2\phi + \delta_3\sin^2\phi\cos^2\phi\right) + 8\frac{\beta_0^2}{\alpha_0^2}\gamma\sin^2\phi$$
$$+ 4\sin\phi\cos\phi(\varepsilon_{26}\sin^2\phi + \varepsilon_{16}\cos^2\phi - \varepsilon_{36})$$
$$V_3 = 2\frac{\beta_0^2}{\alpha_0^2}\gamma\sin^2\phi + 2\varepsilon_{45}\sin\phi\cos\phi$$
$$V_4 = \varepsilon_{15}\cos^3\phi + \varepsilon_{14}\sin\phi\cos^2\phi + \varepsilon_{25}\sin^2\phi\cos\phi + \varepsilon_{24}\sin^3\phi$$
$$V_5 = \varepsilon_{34}\sin\phi + \varepsilon_{35}\cos\phi$$
$$V_6 = \varepsilon_{46}\sin\phi\cos\phi$$
$$V_7 = \varepsilon_{56}\sin\phi\cos\phi$$

with $\varepsilon_{ij} = c_{ij}/c_{33}$; $i = 1, 2, 3, 4, 5$; $j = 4, 5, 6$ and $j > i$.

The parameters ε_{ij} determine the coupling of normal stress with shear strain ($i = 1, 2, 3$) and shear stress with orthogonal shear strain ($i = 4, 5, 6$). For higher-order symmetry of the elastic stiffness matrix, these anisotropic parameters vanish.

B2 *P-SH* reflection coefficients

Here we write the expressions for the anisotropic parts of the linearized reflection coefficients, the corresponding isotropic parts are the same as eq. (B2)

$$^{\mathrm{ani}}R_{PSH}^{\mathrm{ORT}} = \frac{\alpha_0}{\beta_0}\frac{\sin i}{2\cos j\sin(i+j)}\sin\phi\cos\phi\left[(\delta_2 - \delta_1)\sin j - 4\frac{\beta_0^2}{\alpha_0^2}\gamma_1\sin i\cos i\cos j\right.$$
$$\left. + \left(2\left(\varepsilon_1\sin^2\phi - \varepsilon_2\cos^2\phi\right) + (\delta_2 - \delta_1 + \delta_3\cos 2\phi) + 4\frac{\beta_0^2}{\alpha_0^2}\left(\gamma_1 - \gamma_2\sin^2\phi\right)\right)\sin^2 i\sin j\right] \tag{B5}$$

and

$$^{\mathrm{ani}}R_{PSH}^{\mathrm{TRI}} = \frac{\alpha_0}{\beta_0}\frac{\sin i}{2\cos j\sin(i+j)}\left(\sum_{k=1}^{6}\nu_k V_k\right) \tag{B6}$$

where

$$\nu_1 = \sin j$$
$$\nu_2 = \cos j$$
$$\nu_3 = \sin^2 i \sin j$$
$$\nu_4 = \sin^2 i \cos j$$
$$\nu_5 = \sin i \cos i \sin j$$
$$\nu_6 = \sin i \cos i \cos j$$

and

$$
\begin{aligned}
V_1 &= \frac{1}{2}(\delta_2 - \delta_1)\sin 2\phi + \varepsilon_{36}\cos 2\phi \\
V_2 &= \varepsilon_{35}\sin\phi - \varepsilon_{34}\cos\phi \\
V_3 &= \left(\left(\varepsilon_2 \sin^2\phi - \varepsilon_1\cos^2\phi\right) + (\delta_2 - \delta_1 + \delta_3\cos 2\phi) + 4\frac{\beta_0^2}{\alpha_0^2}\left(\gamma_1 - \gamma_2\sin^2\varphi\right)\right)\sin 2\phi \\
&\quad - \varepsilon_{36}\cos 2\phi + \frac{1}{2}(\varepsilon_{26} - \varepsilon_{16})\sin^2 2\phi \\
V_4 &= \varepsilon_{25}\sin^3\phi - \varepsilon_{14}\cos^3\phi + (\varepsilon_{15} - 2\varepsilon_{46})\sin\phi\cos^2\phi + (\varepsilon_{24} - 2\varepsilon_{56})\sin^2\phi\cos\phi - \varepsilon_{35}\sin\phi + \varepsilon_{34}\cos\phi \\
V_5 &= \varepsilon_{46}\sin\phi + \varepsilon_{56}\cos\phi \\
V_6 &= -2\varepsilon_{46}\cos^2\phi.
\end{aligned}
\tag{B7}
$$

Here, we have two S wave splitting parameters, namely

$$
\gamma_1 = \frac{c_{44} - c_{55}}{2c_{55}} \quad \text{and} \quad \gamma_2 = \frac{c_{66} - c_{55}}{2c_{55}}. \tag{B8}
$$