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GEOPHYSICS

Pumping Effect in the Theory of Nonlinear Processes of the Thermal Conductivity Equation Type and Its Application in Geophysics

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Many natural processes are described by nonlinear parabolic equations of the thermal conductivity equation type, in which the coefficient of the medium is a function of the value sought. Although the class of these equations is called equations of thermal conductivity, they are used in the description of absolutely different processes. The general form of these equations in the 1D case can be written as:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[F(T)\frac{\partial T}{\partial x}\right], \tag{1}$$

where coefficient of the medium $F(T)$ has different dependences for various classes of problems. Most frequently, $F(T)$ is described by the power law $F(T) = T^n$. For example, at $n = 1$, Eq. (1) describes the dynamics of free-flow filtration in a porous medium [8]; at $n \geq 1$, gas filtration through a porous medium [6, 13]; at $n = 3$, a thin fluid film flowing downward under the force of gravity [10] and the evolution of long gravity waves of the tidal type in shallow water [3, 4, 11]; and at $n = 6$, the Marshak radiation wave [12].

Many papers were dedicated to the study of self-similar and invariant group solutions of Eq. (1). In particular, Barenblatt [1] obtained self-similar solutions of the first and second kind (incomplete self-similarity) for Eq. (1). However, self-similar solutions are related to the initial problem. In this paper, we shall be interested in the boundary value problem for Eq. (1) without initial conditions.

1. Let us consider a periodical problem for Eq. (1) on a half-line $x > 0$ with boundary conditions

$$T|_{x=0} = f(t), \quad T|_{x\to\infty} < C < \infty, \tag{2}$$

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where $f(t)$ is a periodical function with period τ or frequency $\omega = \frac{2\pi}{\tau}$. In such problems, $f(t)$ usually takes the form $f(t) = T_0 + A\cos\omega t$. This equation can describe fluctuations of water level as it enters a channel, fluctuations of current velocity at the boundary of a porous medium, or fluctuations of temperature at the boundary in the problem of temperature waves. Let us introduce an operator for averaging over period τ:

$$\langle T\rangle = \frac{1}{\tau}\int_t^{t+\tau} T\,dt.$$

Let $\Psi(T)$ be a primitive function of $F(T)$:

$$\Psi(T) = \int F(T)\,dT. \tag{3}$$

We shall assume that $\Psi(T)$ is a single-valued function. Then, we can formulate the following

Statement: Periodical solution of Eq (1) with boundary conditions (2) at $x \to +\infty$ tends to the constant

$$T^{(\infty)} = \Psi^{-1}(\langle\Psi(f(t))\rangle). \tag{4}$$

Indeed, taking into account Eq. (3), we can rewrite Eq. (1)as

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[\frac{d\Psi}{dT}\frac{\partial T}{\partial x}\right] \equiv \frac{\partial^2\Psi}{\partial x^2}. \tag{5}$$

Let us average the left- and right-hand parts of Eq.(5) over period τ. As a result, we get

$$\frac{\partial^2\langle\Psi\rangle}{\partial x^2} = 0, \tag{6}$$

and, consequently, $\langle\Psi\rangle = C_1 x + C_2$. Since $\langle\Psi\rangle$ is nothing other than heat flux averaged over the period, t cannot grow infinitely at $x \to +\infty$; i. e., $C_1 = 0$. It follows from this that $\langle\Psi\rangle = C_2$; i.e., $\langle\Psi\rangle$ is an invariant, which is conserved for all x. As a result, we get

$$\langle\Psi\rangle|_{x=0} = \langle\Psi\rangle|_{x\to+\infty}. \quad (7)$$

At $x \to +\infty$, oscillations attenuate and we shall get at infinity

$$\langle\Psi\rangle|_{x\to+\infty} = \Psi(T^{(\infty)}). \quad (8)$$

Let us denote inverse function to $\Psi(T)$ as Ψ^{-1} and, taking into account (7), we obtain relation (4) from (8). Thus, pure harmonic oscillation of the characteristic of medium T at the boundary of the domain leads to an increase or decrease in its value in the interior of the domain with respect to its mean value at the boundary. Thus, we get the effect of either "pumping in" or "pumping out" of the substance at infinity by a harmonic oscillation at the boundary. This can be called the pumping effect.

In deducing Eq. (6), we assumed that the solution of Eq. (1) with boundary conditions (2) can contain only multiple frequencies divisible by ω. The validity of this assumption can be shown in the case of small values of ratio $\varepsilon = \frac{A}{T_0}$ in the relation for $f(t)$, i.e., at $\varepsilon \ll 1$. Assuming that $F(T)$ is an analytical function, we shall expand it into Taylor series in the neighborhood of T_0. Then, we get Eq. (1) in the following form:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[\left(F(T_0) + \frac{dF(T_0)}{dT}\varepsilon T + \frac{1}{2}\frac{d^2F(T_0)}{dT^2}\varepsilon^2 T^2 + \dots\right)\frac{\partial T}{\partial x}\right]. \quad (9)$$

We seek the solution of Eq. (9) in the form of asymptotic series with respect to ε:

$$T = T^{(0)} + \varepsilon T^{(1)} + \varepsilon^2 T^{(2)} + \dots \quad (10)$$

Substituting (10) into (9) and gathering terms of the zero, first, etc., orders by ε, we obtain a system of reduced inhomogeneous linear equations of thermal conductivity, which would contain only multiple harmonics.

2. It is easy to find the value of invariant $\langle\Psi\rangle$ at infinity, because oscillations attenuate there. However, in practice, such a problem occurs frequently for limited regions and when the problem is formulated over a limited segment $0 \le x \le L$, the procedure of finding the invariant considered in the previous section cannot be repeated at $x = L$. In the general case, solution of Eq. (1) over a segment can be solved only numerically. However, if ratio $\varepsilon = \frac{A}{T_0}$ in the relation for $f(t)$ is a small value (i.e., $\varepsilon \ll 1$), it is possible to find an analytical expression for the pumping effect at the other end of the segment at $x = L$. Thus, let us consider Eq. (9) and limit the expansion of $F(T)$ only by the terms of the first order with respect to ε:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[(\alpha + \beta\varepsilon T + O(\varepsilon))\frac{\partial T}{\partial x}\right], \quad (11)$$

where $\alpha = F(T_0)$, $\beta = \frac{dF(T_0)}{dT}$. At the right end of the segment, we specify the boundary condition of the second kind

$$\left.\frac{dT}{dx}\right|_{x=L} = 0, \quad (12)$$

which corresponds physically to the condition of a rigid wall in the problem of the evolution of long waves in shallow water or lack of thermal flux in thermal conductivity problems. We seek the solution of Eq. (11) with boundary conditions (2), (12) in the form of asymptotic expansion (10) with boundary conditions

$$T^{(0)}|_{x=0} = A\cos\omega t, \quad \left.\frac{\partial T^{(0)}}{\partial x}\right|_{x=L} = 0, \quad T^{(1)}|_{x=0} = 0,$$

$$\left.\frac{\partial T^{(1)}}{\partial x}\right|_{x=L} = 0.$$

We seek the solution for the first approximation $T^{(0)}$ as

$$T^{(0)} = \mathrm{Re}(Q(x)e^{i\omega t}) = \frac{Q(x)e^{i\omega t} + Q^*(x)e^{-i\omega t}}{2}, \quad (13)$$

where Re denotes the real part and the asterisk denotes a complex conjugated function.

Substituting (13) into the first approximation of Eq. (11), we get the solution for

$$Q(x) = A\frac{\cosh[\lambda(L-x)]}{\cosh(\lambda L)}, \quad (14)$$

where $\lambda = (1+i)\sqrt{\frac{\omega}{2\alpha}}$. Substituting (14) into (13) and then into the second approximation of Eq. (11) with respect to ε, we obtain a solution for $T^{(1)}$ containing a periodical part and time-independent additive; this describes the pumping effect:

$$T^{(+)}(x) = -\frac{\beta}{4\alpha}[Q(x)Q^*(x) - Q(0)Q^*(0)]. \quad (15)$$

Equation (15) gives a quantitative value of the pumping effect at point x. At the end of segment $x = L$, the pumping effect would be

$$T^{(+)}(L) = -\frac{\beta A^2}{4\alpha}\left[\frac{1}{\cosh(\lambda L)\cosh(\lambda^* L)} - 1\right]. \quad (16)$$

At $L \to \infty$, we get

$$T^{(+)}(\infty) = \frac{\beta A^2}{4\alpha}. \quad (17)$$

Equation (16) allows us to estimate distance $L_{(+)}$, where the mean temperature approaches asymptotic solution (17):

$$L_{(+)} = [2\mathrm{Re}(\lambda)]^{-1} = \left(\frac{\alpha}{2\omega}\right)^{1/2}.$$

In practice, medium function $F(T)$ in Eq. (1) is often a linear function (propagation of temperature waves in water, ice, and soil) $F(T) = \alpha + \beta T$. In this case, we have the following relation for the pumping effect:

$$T^{(\pm)} = -b \pm \sqrt{b^2 + \frac{A^2}{2}}, \quad b = \frac{\alpha}{\beta} + T_0. \tag{18}$$

If $b < 0$, one should take a minus sign in Eq. (18); if $b > 0$, a plus sign is taken; and if $\frac{A}{b} \ll 1$ and $\frac{\alpha}{\beta} \gg T_0$, Eq. (21) is simplified and reduced to Eq. (17).

3. *Burgers equation.* Fluid motion in a thin layer of saturated ground with a lower and upper inflow of fluid, as well as flows in perforated tubes with lateral inflow, are described by the Burgers equation [9]. This equation is also valid for the pumping effect. Let us consider a periodical problem for the Burgers equation over a semi-infinite straight line:

$$\varsigma_t + (\varsigma_x)^2 = \nu\varsigma_{xx} \tag{19}$$

with boundary conditions

$$\varsigma|_{x=0} = \varsigma^{(0)}\sin\omega t, \quad \varsigma|_{x\to\infty} < C < \infty. \tag{20}$$

The Hopf–Cole substitution $\varsigma = 2\nu\ln\varphi$ reduces Eq. (19) to a linear equation of thermal conductivity, whose solution, taking into account boundary conditions (20), is written as

$$\varphi(x, t) = \frac{2}{\pi}\int_0^\infty \exp\left\{-\xi^2 - \frac{\varsigma^{(0)}}{2\nu}\sin\left[\omega\left(t - \frac{x^2}{4\nu\xi}\right)\right]\right\}d\xi. \tag{21}$$

Let us use generating function for the Bessel functions of the first kind J_n to calculate integral (21). Then, we can write

$$\varphi(x, t) = \sum_{n=-\infty}^{+\infty} \frac{2\exp(in\omega t)}{\sqrt{\pi}} J_n\left(-\frac{\varsigma^{(0)}}{2\nu i}\right) \times \int_0^\infty \exp\left[-\xi^2 - \frac{in\omega x^2}{4\nu\xi^2}\right]d\xi. \tag{22}$$

In order to find the limit of function $\varphi(x, t)$ in (22) at $x \to +\infty$, let us use the Riemann–Lebesgue lemma about integrals of oscillating functions [7]. As a result, we obtain

$$\lim_{x\to+\infty}\varphi(x, t) = \frac{2}{\pi}J_0\left(-\frac{\xi^{(0)}}{2\nu i}\right)\int_0^\infty \exp(-\xi^2)d\xi = J_0\left(-\frac{\xi^{(0)}}{2\nu i}\right) = I_0\left(\frac{\xi^{(0)}}{2\nu}\right), \tag{23}$$

where I_0 is Bessel function of an imaginary argument. Returning to the initial function $\varsigma(x, t)$, we obtain from (23) the relation for the pumping effect in Burgers equation (19).

4. APPLICATION OF PUMPING EFFECT TO GEOPHYSICAL PROCESSES

4.1. *Increase of mean tidal level in shallow water* [4]. Transformation of tidal wave with period T, when it reaches shallow water and depth h does not exceed the Stokes layer thickness $h_{st} = \sqrt{AT}$ (A is kinematic coefficient of vertical turbulent exchange), is described by nonlinear parabolic equation [3, 4] for ζ level elevation

$$\zeta_t = \frac{g}{3A}\nabla[(h+\zeta)^3\nabla\zeta], \tag{24}$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is the Hamilton operator and g is acceleration due to gravity.

The 1D analog of Eq. (24) would have a form similar to (1) with $F(\zeta) = \frac{g}{3A}(h+\zeta)^3$. Fluctuations in level at the offshore boundary of the shallow zone are determined by incident tidal wave $\zeta = \zeta_0\sin\omega t$, $\omega = \frac{2\pi}{T}$. At $h = \mathrm{const}$, we get the following value of the invariant:

$$(h + \zeta^{(+)})^4 = \int_0^T (h + \zeta_0\sin\omega t)^4 dt. \tag{25}$$

The level increase at infinity is obtained by calculating the integral in (25):

$$\zeta^{(+)} = h\left\{\left[1 + 3\left(\frac{\zeta_0}{h}\right)^2 + \frac{3}{8}\left(\frac{\zeta_0}{h}\right)^4\right]^{1/4} - 1\right\}. \tag{26}$$

In the case of small values of the ratio of the amplitude of the incident tide ζ_0 to depth h, i.e., $\varepsilon = \frac{\zeta_0}{h} \ll 1$, Eq. (26) can be simplified:

$$\zeta^{(+)} \approx \frac{3}{4}\frac{\zeta_0^2}{h}.$$

Thus, the mean level increases as the coast is approached. In places of wave attenuation, we get a stationary level increase. At the mouths of rivers flowing into a tidal sea, the pumping effect leads to the displacement of the headwater further to the mouth of a river.

4.2. *Intrusion of marine waters into the mouths of rivers and underground levels in tidal seas.* Saline seawater intrudes the mouths of rivers at bottom levels as a halocline. The dynamics of a halocline within a two-layer fluid in a channel of constant depth H is described by a system of two nonlinear parabolic equations [2] for the free surface $z = \varsigma(t, x)$ and interface between layers $z = \eta(t, x)$:

$$\frac{\partial\varsigma}{\partial t} = \frac{g}{3A}\frac{\partial}{\partial x}\left\{(H-\varsigma)^3\frac{\partial\varsigma}{\partial x}\right.$$

$$\left.+\frac{\Delta\rho}{\rho_1}\left[(H-\eta)^3+\frac{3}{2}(H-\eta)^2(\eta-\varsigma)\right]\frac{\partial\eta}{\partial x}\right\}, \quad (27)$$

$$\frac{\partial\eta}{\partial t} = \frac{g}{6A}\frac{\partial}{\partial x}\left\{\frac{2\Delta\rho}{\rho_1}(H-\eta)^3\frac{\partial\eta}{\partial x}\right.$$

$$\left.+[2H^3-3H^2\eta+\eta^3-3\varsigma(H-\eta)^2]\frac{\partial\varsigma}{\partial x}\right\}, \quad (28)$$

where A is the kinematic coefficient of vertical turbulent exchange, g is acceleration due to gravity, $\Delta\rho$ is the difference between sea and river water, and ρ_1 is density of river water.

In the mouths of rivers flowing into tidal seas, the pumping effect in Eq. (27) for the free surface $z = \varsigma(t, x)$ leads to the displacement of headwater further to the mouths of rivers. The pumping effect in Eq. (28) for the interface between layers $z = \eta(t, x)$ leads to an extension of the halocline and the consequent deeper penetration of saline water into the mouths of tidal rivers. Numerical experiments demonstrated [5] that an increase in the distance of saline water penetration in the mouth of a tidal river (relative to a similar mouth of a nontidal river) can reach several kilometers.

A similar effect is manifested during intrusion of seawater into underground levels in tidal seas, which leads to distant penetration of the halocline along the aquifer. It is interesting that in the case of water-saturated ground with upper or lower inflows, in which fluid motion is described by the Burgers equation, the pumping effect during harmonic oscillation of current velocity at the lateral boundary of the layer (for example, due to tidal oscillations of pressure) leads to the formation of permanent inflow, which is directed deeper from the outer boundary.

4.3. *Effect of mean temperature decrease in ice, glaciers, and permafrost.* The coefficient of thermal conductivity for freshwater ice is specified by linear function of temperature $F(T) = 5.35 \cdot 10^{-3}(1–4.8 \cdot 10^{-3}T)$ cal/(°C cm s). In this case, Eq. (17) can be applied. As a result, we get

$$T^{(-)} = \frac{\beta A^2}{4\alpha} = -1.2 \cdot 10^{-3}A^2.$$

Thus, a negative pumping effect is observed in ice (including glaciers and permafrost); i.e., heat is pumped out from the lower layers of the ice at increasing amplitude of temperature fluctuations at the upper boundary of the ice massif. In the permafrost zone, this effect can reach significant values. For example, the amplitude of annual oscillations of air temperature in Yakutia can reach $A = 40$°C, and $T^{(-)} = –1.9$°C at such amplitudes.

Thus, we have shown that the pumping effect, manifested in many geophysical processes, plays an important role in heat and energy redistribution on the earth.

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