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GEOPHYSICS

Construction of Analytical Approximations of Geopotential Fields with the Consideration of Their Fractal Structure

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Principal trends in the development of mathematical geophysics at the modern stage include the comprehensive development of the approximation approach [1]. Researchers widely use approximations of external elements of gravitational and magnetic fields by equivalent source assemblages (*source-like approximations* [2]) and *numerical field simulations* [3]. We propose a new method for the approximation of geopotential fields by grid distribution of sources. This method takes into account the fractal structure of the fields and is based on the quadrotree technique used for the compression of digital graphic images [4].

The essence of the considered approximation transformations is as follows: all information related to the observed geophysical field $U(x, y, z)$ is stored as a certain number k of vectors of parameter $\mathbf{P} = \{p_1, p_2, \ldots, p_n\}$ of sources creating the simulation field $U^{\text{mod}}(x, y, z)$, which is practically equivalent to the field $U(x, y, z)$. The unknown parameters of sources are found by solving an inverse problem (IP) usually consisting in minimization of the functional

$$F(k, \mathbf{P}) = \sum_{i=1}^{m} [U(x_i, y_i, z_i) - U^{\text{mod}}(x_i, y_i, z_i, k, \mathbf{P})]^2$$

within the ensemble μ of the field specification points. In the process of minimization, we solve a system of linear algebraic equations (SLAE) for the linear IP formulation and a sequence of SLAE for the nonlinear IP formulation. At $F(k, \mathbf{P}) \le \varepsilon$, where ε is a sufficiently small value comparable with the measurement accuracy, the IP solution is considered to be attained.

The greatest difficulties in developing analytical approximations of the field $U(x, y, z)$ are related to large and superlarge dimensions of SLAE [1, 2], since $\mu \ge 10^5$–10^6 when many practical problems are solved. Consequently, in order to create efficient computational algorithms implementing a *sourcelike approximation*, it is necessary to use a minimal number k of field sources $U^*(x, y, z)$ and reduce the number n of the vector $\mathbf{P}$ parameters.

There are a number of approaches to this problem: for instance, development of an approximation construction represented by fixed-geometry point masses located under each field specification point [5, 6] or a set of 3D singular sources with their parameters defined by the nonlinear programming method [7]. However, in the former instance, the number of field sources $k = \mu$ is excessive. In the latter case, a manual modeling of the initial spatial distribution of disturbing objects is required, and a relatively complicated nonlinear IP is solved.

It is well known that the spatial distribution of gravitational and magnetic field anomalies has an approximate scale invariance [8]. Geopotential fields are multifractals, because they have a self-similar hierarchically ordered structure that can be taken into consideration in the process of approximation.

It is possible in one way or another to compare the spatial distribution of equivalent sources with characteristic features of the approximated field $U(x, y, z)$ revealed in particular scales of its investigation. It is expedient to start building the approximation construction from the roughest (small-scale) field approximation and maximum source location depths, i.e., to carry out simulation of the highest-energy regional field component and then gradually proceed to a detailed field analysis and modeling of local anomalies of various orders in larger scales.

Thus, the scale-invariant irregular features of the simulated field will correspond to scale-irrelevant frag-

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ments of the 3D model of sources that will be a multifractal as well.

The principal concept of the quadrotree technique consists in representation of the initial field specification region $S = S(x, y)$ in the form of nonoverlapping subregions S_α: $S = \cup_\alpha S_\alpha$, $\cap_\alpha S_\alpha = \varnothing$, the number α of which is growing in geometrical progression with transition from a smaller to a larger scale (higher detailing). Let us further call the subregions S_α as rank blocks and the process of their successive diminution as an increase in the quadrotree depth N. Distribution of equivalent sources can be related to splitting the region S into rank blocks controlled, in turn, by morphologic features of the approximated field.

Let us examine the algorithm of the approximation of a gravitational field with its values specified on the physical surface of the earth within a square region S with the side L in the nodes of a regular grid. The initial data are two square matrices containing the values of the observed gravitational field $\mathbf{\Delta g} = \{\Delta g_{ij}\}$ and the observation surface elevations $\mathbf{H} = \{H_{ij}\}$, $1 \le i \le m$, $1 \le j \le m$, $\mu = m^2$.

Initially, the region S is split into four rank blocks S_1 of the first level and has a square shape with the side $l_1 = \frac{L}{2}$ (the subindex corresponds to the quadrotree depth). Point sources (spheres) are distributed in the rank block centers at a depth of $l_1 \le h_1 \le 2l_1$ from the earth's surface. The field Δg_1 values are calculated by averaging the respective values within the rank blocks and are related to their centers. Values of the height H_1 in the centers of the blocks are calculated by bilinear interpolation on the nearest four points. The masses M_1 of the four spheres are determined by solving the SLAE $GM_1 = \mathbf{\Delta g}_1$, where G is the solution operator for the direct gravimetric problem for a sphere with $M_1 = 1$. The rather sufficient SLAE conditionality is ensured by the above relation $\frac{l_1}{h_1}$. In all the μ points of the initial field specification, we calculate the model field $\mathbf{\Delta g}_1^{\text{mod}}$ related to sources with the known masses M_1. The difference of the observed and model fields $\mathbf{\Delta g}_1^* = \mathbf{\Delta g} - \mathbf{\Delta g}_1^{\text{mod}}$ substitutes for the initial field $\mathbf{\Delta g}$ at the second level of the quadrotree depth.

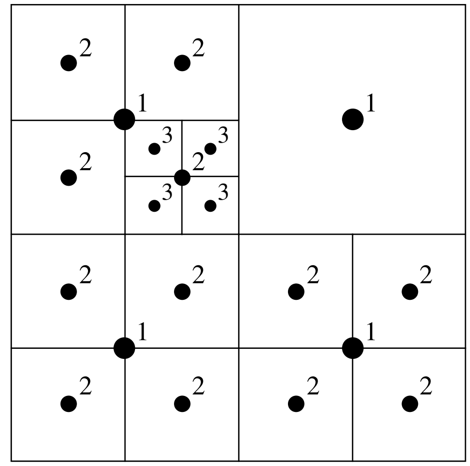

Fig. 1. Scheme of region S splitting into rank blocks for the quadrotree depth $N_{\max} = 3$. (1) Initial field specification points; (2) equivalent sources (figures correspond to splitting levels).

Further, the quadrotree depth is increased: each rank block S_N is split into four smaller square blocks S_{N+1} with sides $l_{N+1} = \frac{l_N}{2}$. However, if within any rank block S_{N+1}^k, $1 \le k \le 4^N$, we have attained the required accuracy of approximation of the field $\sqrt{\sum_1^{\mu_N} (\Delta g^*)^2 \mu_n^{-1}} \le \varepsilon$, , where $\mu_N = \frac{\mu^2}{4^N}$ is the number of the field points within the block, the source is not placed in the block center (Fig. 1). Then, we reiterate all of the above described procedure of determination of equivalent source masses $\mathbf{M}_{N+1}$ and the calculation of the difference field $\mathbf{\Delta g}_{N+1}^*$.

Parameters characterizing the approximation of the Δg field in the Yuryuzan–Sylva Depression

Quadrotree depth N	Number of field Δg_N points	Source location depth h_N, km	Number of sources k	Approximation accuracy ε, mGal	Ratio $\frac{k}{\Delta g_N}$, %
1	16	24.0	16	±2.54	100
2	64	12.0	60	±1.00	93.7
3	256	6.0	228	±0.51	89.0
4	1028	3.0	668	±0.27	65.0
5	4096	1.5	1790	±0.13	43.7
6	16384	0.75	2791	±0.06	17.0

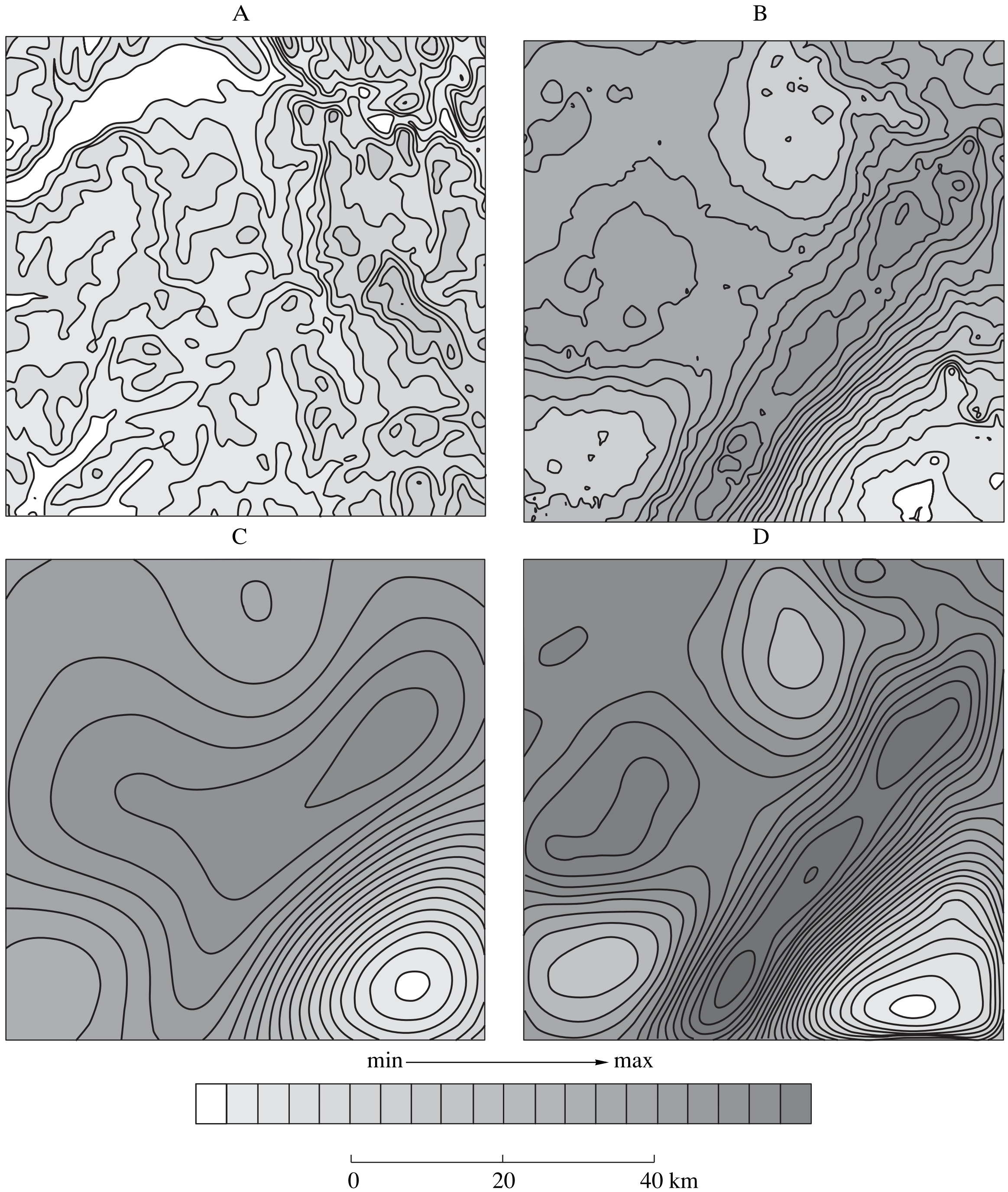

Fig. 2. Initial data and transformers of the gravitational field of the Yuryuzan–Sylva Depression. (A) Terrain contour map; (B) observed Bouguer gravity field Δg; (C) the upper half-space field Δg at level $H = 10$ km; (D) vertical derivative V_{zz} at level $H = 5$ km

The iteration process of the development of approximation construction is completed when the a priori specified mean-square approximation error ε is attained in all the field specification points μ or when the maximum quadrotree depth $N_{\max}$ is reached and the rank block size $l_{N_{\max}}$ begin to match the internode interval of the initial field matrix $\frac{L}{m-1}$.

Thus, the source distribution network is adjusted to morphological features of the initial gravitational field. The resulting model is a set of k balls located at several depth levels $h_1, h_2, \ldots, h_{N_{\max}}$ with compact horizontal distribution of sources near regions of the most complicated field pattern.

The explained algorithm can be upgraded for operation with data defined within regions of casual geometry in irregular network nodes. For this purpose, when determining values of Δg_i, $i = 1, 2, \ldots, N_{\max}$ in the centers of rank blocks, it is sufficient to use interpolation (for instance, by the weighted distance method) instead of averaging. In this case, the choice of interpolation method is not important, because, irrespective of the adopted method of the initial field Δg decomposition in "variable scale" components, the quality of problem solution in the initial field points is controlled by the value $\varepsilon = \sqrt{\sum_{i=1}^{\mu}\sum_{j=1}^{N_{\max}}(\Delta g_{ij} - \Delta g_{ij}^{\text{mod}})^2\mu^{-1}}$.

Let us examine a model example demonstrating the scope of the algorithm. The field Δg is a gravitational effect of seven balls located at different depths. The size of the initial matrix $\mathbf{\Delta g}$ is 128 rows and 128 columns ($\mu = 16{,}384$); the field Δg variation range is –10 to 19 mGal. Application of the above algorithm yields a grid model incorporating 1013 balls, ensuring the mean-square discrepancy $\varepsilon = \pm 0.019$ mGal for the fields Δg and Δg^*. As it is obvious from the example, the number of equivalent sources in the model is one order of magnitude smaller than the number of the initial field specification points.

The quadrotree technique was used for interpretation of the Bouguer gravity field over a large tectonic structure (Yuryuzan–Sylva Depression) with petroleum potential. The depression is situated in the Ural Foredeep. The study region was characterized by the following parameters: area ~ 4000 km^2; altitude 100–450 m (Fig. 2A); number of the initial field points $\mu = 16384$; and variation range of Δg values >35 mGal (Fig. 2B).

Application of the algorithm yielded a high accuracy of the observed field approximation $\varepsilon = \pm 0.06$ mGal based on 5553 sources (table). Figures 2C and 2D show the gravitational field transformers calculated with the application of respective solution operators for the direct gravimetric problem from the developed approximation structure [5, 6].

The proposed approximation method has the following advantages:

(i) decomposition of a problem in the form of sequential solution of a SLAE series of relatively small dimensionality;

(ii) automatic location of sources without cumbersome computational works; and

(iii) development of an approximation structure characterized by a small number of parameters ($\mathbf{P} = \{x, y, z, M\}$, where x, y, z are coordinates of centers of gravity; M are ball masses; $k \ll \mu$; and μ is the number of field points) ensuring a high match accuracy ε of the observed and simulated fields.

Especially interesting is the application of the algorithm for the compilation of data bases in geoinformation systems. In addition to an array of observed field values, we propose to save an approximation of the $k\mathbf{P}$ type (with a significantly smaller dimensionality). This will make it possible to restore the field in arbitrarily selected points in the space beyond the sources and to carry out asymptotically optimal accuracy transformations that take into account the pattern of observation surface topography $H = H(x, y)$.

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