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GEOPHYSICS

Application of High-Order Spectra and Coherence for Analyzing Nonlinearity of a Ground Response

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It is known that high-order spectra (second order) and coherences (greater than two) are used to detect phase-correlated oscillations in stationary time series. High-order coherences (HOCs) serve as a measure of phase relations [1]. HOC functions are determined as high-order spectra (HOS) normalized by power spectra. The HOCs (mainly, bicoherence) are more frequently used in practical applications: estimates of bicoherence obtained for time series are compared with expected mean values for the case when the true value of bicoherence is zero (for example, for Gaussian noise). High bicoherence of the signal at the output of any system indicates that the signal contains components with a quadratic relation in phase, probably caused by the quadratic nonlinearity of the system.

In geophysics, HOC and HOS have not received wide application yet. Bispectral analysis was used to study nonlinear effects in low-frequency oceanic waves [2]. Based on investigation of the presence of nonlinearly constrained components in seismic noise and coda-waves, the author of [3] concluded that the obtained estimates of bicoherence point (within the errors of calculations) to linearity of the seismic noise and a high content of nonlinearly constrained components in coda-waves. This is, however, explained not by nonlinear effects, but by the nonstationary properties of coda-waves. In [4], bispectral analysis was applied to study the fine structure of the Earth's free oscillations induced by the Chilean and Alaska earthquakes. The obtained results point to phase correlation between different low-frequency modes of oscillations. Based on the method of bispectral analysis, the author of [5] distinguished a correlation between tides and random local variations in the sea level.

The advent of powerful computers widened the capabilities of using mathematical software of HOS and HOC to analyze stationary time series. In [6], the bicoherence of seismic noise was estimated on the basis of long-wave records in order to increase the accuracy; estimates of bicoherence of seismic noise recorded at cliff rocks and on soft ground were compared; and differences related to nonlinear response were not found within the errors of calculations. High values of bicoherence recorded at the frequencies of technogenic harmonics indicate that industrial sources emit oscillations not only at the main frequency, but also at its higher harmonics. These frequencies are not always phase-correlated, because they are emitted by the same source. Hence, HOS and HOC can also be used to distinguish oscillations in time series related to one source at frequencies f, $2f$, $3f$, $4f$, $5f$, and so on. Since HOS and HOC possess the property of noise suppression (because the signal is a non-Gaussian process and additive noise is close to the Gaussian process, for which all HOS and HOCs are identical to zero), they can be used to detect even weak (in amplitude) higher harmonics of the main frequency over the noise background.

The obtained estimates of bicoherence of seismic noise do not confirm the presence of nonlinear components related to quadratic nonlinearity of the medium. The author of [7] showed that grounds are predominantly characterized by nonlinearities of odd types. Hence, detection of nonlinearly constrained components in seismic noise requires, first of all, investigation of tricoherence (coherence of the 4th order) and coherence of the 6th order.

The present paper is devoted to numerical experiments of bicoherence, tricoherence, and coherences of the 5th and 6th orders of noise and monochromatic signals that passed through a nonlinear medium (a horizontally layered ground section). We demonstrate the possibility of applying HOC to investigate the nonlinear response of the ground.

If nonlinear transformations are spectrally uniform (which is usually satisfied), diagonal HOS and HOC values give an idea about the HOS and HOC values in other spectral ranges. In many cases, this fact allows us

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to use without significant losses a clearer presentation of one-dimensional HOS and HOC instead of multidimensional ones. One-dimensional diagonal ($\omega_1 = \omega_2 = \ldots = \omega_{N-1} = \omega$) values of coherences of the 4th, 5th, and 6th orders are composed similarly to bicoherence $r_3^2(\omega_1, \omega_2)$:

$$r_3^2(\omega_1, \omega_2) = \frac{|C_3(\omega_1, \omega_2)|^2}{C_2(\omega_1)C_2(\omega_2)C_2(\omega_1 + \omega_2)}.$$

Tricoherence is calculated from the formula

$$r_4^2(\omega) = \frac{|C_4(\omega)|^2}{C_2^3(\omega)C_2(3\omega)},$$

coherence of the 5th order is calculated from the formula

$$r_5^2(\omega) = \frac{|C_5(\omega)|^2}{C_2^4(\omega)C_2(4\omega)},$$

and coherence of the 6th order is calculated from the formula

$$r_6^2(\omega) = \frac{|C_6(\omega)|^2}{C_2^5(\omega)C_2(5\omega)}.$$

In order to estimate HOC, a stationary time series $\{x_k\}$ is divided into p equal nonoverlapping intervals, from which HOC values are calculated and averaging is performed. The estimates become smooth at sufficiently large p [1, 3]. The author of [3] showed that the expected mean value of bicoherence is $\frac{1}{p}$; at large p, the squared module of bicoherence is characterized approximately by χ^2-distribution with two degrees of freedom, and the 95% confidence level is approximately equal to $3/p$. Since HOCs are constructed similarly to bicoherence, it is easy to show that squared modules of tricoherence and coherences of the 5th and 6th orders are also characterized by χ^2-distribution with two degrees of freedom, and the mean value for the diagonal terms is $\frac{(N-1)!}{p}$, where n is the order of coherence, and multiplier $(N-1)!$ appears due to symmetrical properties of HOC. Mean modules of HOC would be higher than $\frac{(N-1)!}{p}$ for a Gaussian process that passed a nonlinear system and was enriched with combination harmonics. High values of tricoherence (coherences of the 5th and 6th orders) would indicate the presence of cubic nonlinearity (nonlinearities of the 4th and 5th orders) in the system and give their quantitative estimates. Thus, based on estimates of HOC of the output signals, one can determine the types of system nonlinearity and obtain its quantitative estimates.

We investigated the HOC of monochromatic and noise signals at the output of the horizontally layered medium of 200-m-thick alluvial rocks (overlying the crystalline basement) without seismic boundaries and ground waters. They are characterized by a constant (in depth) density of 2.05 g/cm^3 but a gradual increase in the velocity of S waves and maximal shear stress τ_{max} with depth. We preliminarily calculated the vertically upward propagation (one-dimensional approximation) of seismic waves in the ground profile (a classical problem of engineering seismology) using the author's software based on an algorithm described in detail in [11], which also describes the studied ground profile.

We considered the following two cases: (1) the input signal is monochromatic; (2) the Gaussian noise occupies a wide range of frequency (~0.1–170 Hz). Figures 1 and 2 demonstrate the forms of input (*1*) and corresponding output (*2*) signals for two different intensities of the input signal, the spectra of input and output signals (*3*), and the diagonal HOC values (*4–7*). Increasing the intensity of the input signals increases their nonlinear distortions, leading to changes in the amplitude and spectral composition of oscillations at the surface. The spectra of input signals are shown with thin lines; the corresponding spectra of output signals, with heavy lines. If the input signal is monochromatic (Fig. 1a), its nonlinear distortions in the medium are caused by amplification of the 3rd, 5th, 7th, …, and other odd divisible harmonics of the main frequency, suggesting odd types of ground nonlinearity. Nonlinearities of odd types are also characterized by the effects of self-influence, resulting in the formation of harmonics of the initial frequencies [12].

The power spectra and HOCs of input and output signals quantitatively characterize nonlinear distortions of signals in the medium. The figures show squared modules of bicoherence $r_3^2(\omega)$ (4), tricoherence $r_4^2(\omega)$ (5), and coherences of the 5th $r_5^2(\omega)$ (6) and 6th $r_6^2(\omega)$ (7) orders. Thin lines show the HOC of the input signals; heavy lines, the corresponding HOC of the signals recorded at the surface. It is seen from Fig. 2b that an increase in the intensity of the input noise signal leads to an increase in tricoherence and coherence of the 6th order of the signals at the surface. If the input signal is monochromatic (Fig. 1), the increase in coherences of the 4th and 6th orders at the frequency of the input signal (5 Hz) points to the generation of the 3rd and 5th divisible harmonics of the main frequency (5 Hz). When the amplitudes of the input monochromatic signal are large, the coherences of the 4th and 6th orders at the frequency of the input signal are close to unity. This means that a 100% phase correlation exists between the harmonic of the main frequency and its 3rd and 5th harmonics. The 3rd and 5th harmonics are a result of nonlinear distortions of the main frequency harmonic. We note a high sensitivity of HOC (especially, tricoherence and coherence of the 6th order) to the appearance of

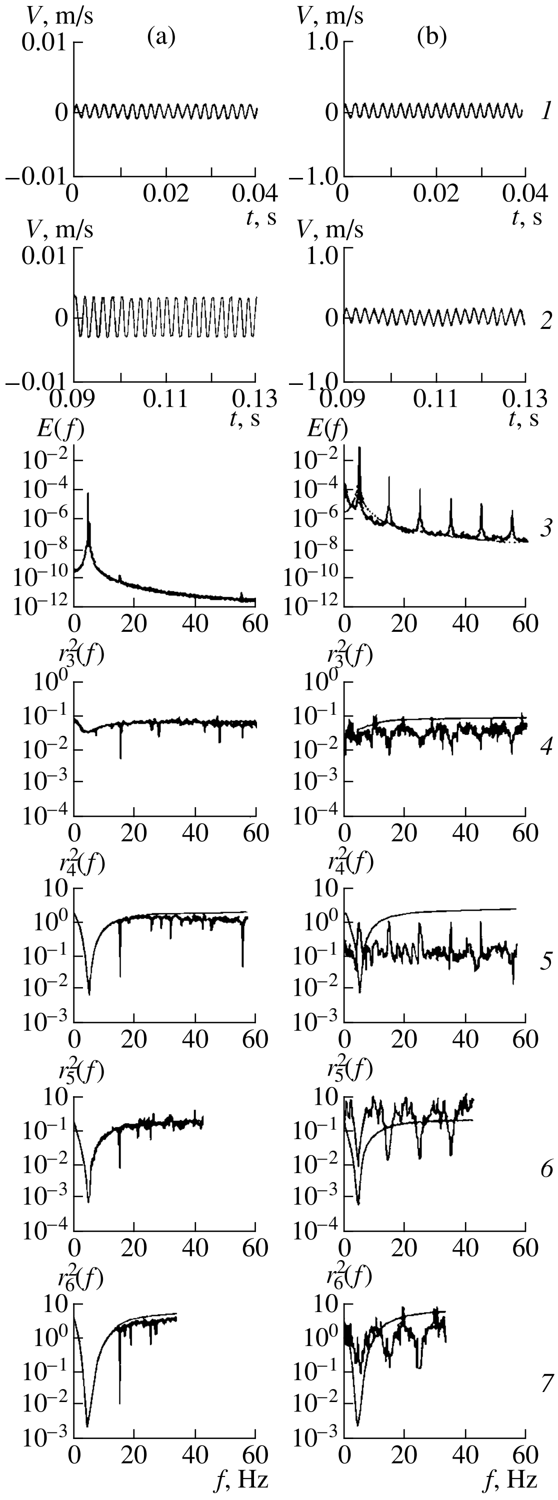

Fig. 1. Variations in the form, spectra, and HOC of harmonic signals with a frequency of 5 Hz and amplitudes of (a) 0.001 and (b) 0.1 m/s, which passed a ground section. Notations here and in Fig. 2: (*1*) input signal; (*2*) response of the ground; (*3*) power spectrum of the input (thin lines) and output (heavy lines) signals; (*4–7*) one-dimensional functions: bicoherence $r_3^2(\omega)$, tricoherence $r_4^2(\omega)$, coherences of the 5th $r_5^2(\omega)$ and 6th $r_6^2(\omega)$ orders of input (thin lines) and output (heavy lines) signals.

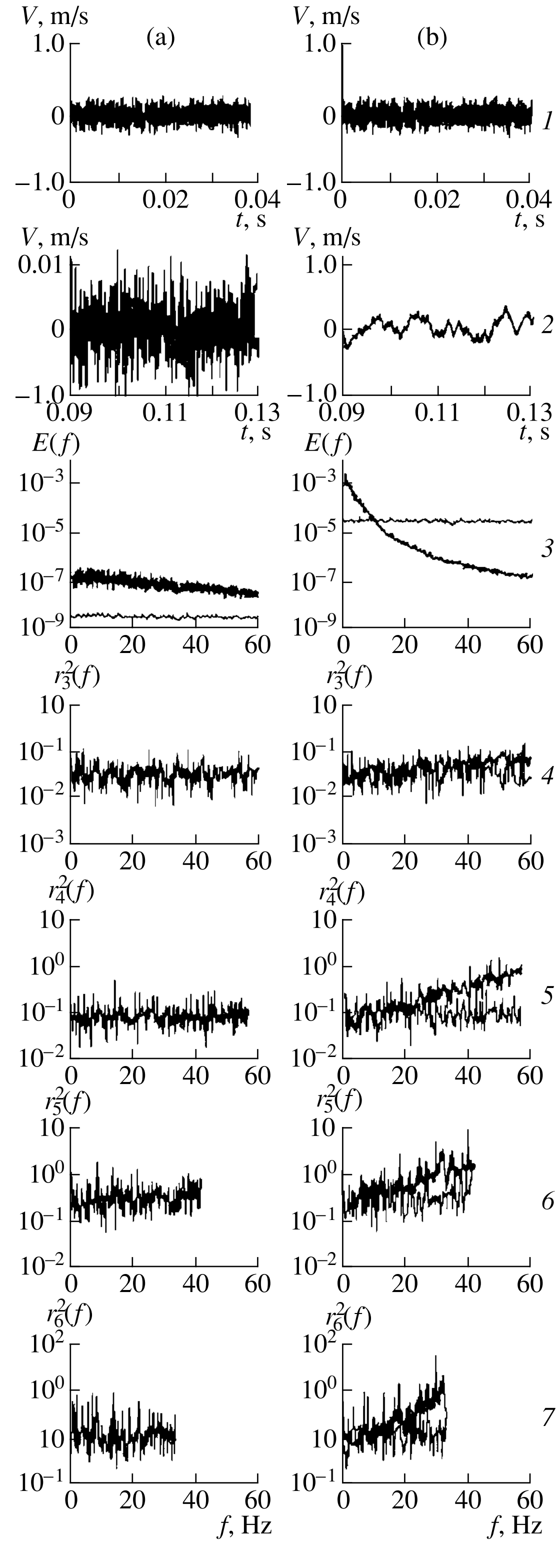

Fig. 2. Variability of the forms, spectra, and HOC of the Gaussian noise, which passed the ground section, in a wide range of frequencies with the r.m.s. amplitude $\sqrt{\langle V^2(t)\rangle}$ equal to (a) 0.001 and (b) 0.1 m/s.

Squared modules of HOC oscillations at the surface for the input signal of Gaussian noise average within the spectral band

$\sqrt{\langle V^2(t)\rangle}$, m/s	$\langle r_3^2(\omega)\rangle$	$\langle r_4^2(\omega)\rangle$	$\langle r_5^2(\omega)\rangle$	$\langle r_6^2(\omega)\rangle$
0.001	0.037	0.105	0.38	1.8
0.005	0.04	0.107	0.43	1.72
0.02	0.041	0.119	0.48	2.12
0.1	0.066	0.35	0.99	4.49
$\frac{(N-1)!}{p}$	0.036	0.11	0.43	2.1

combination harmonics even at weak manifestations of nonlinearity (Fig. 1a).

If the amplitudes of the input signals are sufficiently large, the propagation of these signals in nonlinear medium is accompanied by strong nonlinear distortions. The spectral density of power tends to transform to $E(f) \sim f^{-k}$ regardless of the spectral composition of the input signal (Figs. 1b, 2b). The mean values of squared HOC modules approach $\frac{(N-1)!}{p}$, which is characteristic of noise signals, and exceed it. The table presents the mean values of squared HOC modules (averaged within the spectral band) of the Gaussian noise signals at the output of the ground profile at different intensities of input signals. For comparison, the corresponding values of $\frac{(N-1)!}{p}$ ($p = 55$) are given in the lower row. It is seen from the data presented in the table and Fig. 2 that, when the input signal of Gaussian white noise is sufficiently intense, the squared HOC modules exceed the values of $\frac{(N-1)!}{p}$, especially at high frequencies.

The HOC values demonstrate the prevailing share of cubic nonlinearity in the response of the ground. However, the contribution of nonlinearities of the 4th and 5th orders is also high. It is known that the nonlinearity type of the ground response is determined by the form of stress–strain hysteresis dependence in ground layers [7]. At high intensities of the input signal, the functions describing the branches of loading (or unloading) of stress–strain dependence can be presented as comparable (in amplitudes) odd and even functions of the ground response. At low intensities of the input signal, the stress–strain dependence functions are predominantly odd, and nonlinear components of odd types dominate in the ground response.

It is seen from Fig. 2 that the sensitivity of HOC to weak manifestations of nonlinearities in the ground response is not high in the case of noise signals. Thus, it is not easy to detect nonlinear components in real seismic noise. In principle, the HOC makes it possible to detect them. However, detection and quantitative estimation of nonlinear components in the seismic noise require the following conditions: (1) selection of stationary records of long duration for the analysis; (2) estimation of not only bicoherence, but also other HOCs (tricoherence and coherences of the 5th and 6th orders); and (3) analysis of a number of parameters (squared HOC modules, their mean values in the spectral band and the high-frequency range, the number of values exceeding the 95% and 99% confidence levels, deviations of the distribution of squared HOC modules from the χ^2-distribution, and others).

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