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Runup of Nonlinearly Deformed Waves on a Coast

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The problem of runup of long nonbreaking waves on a plane slope is well developed from the mathematical point of view within the framework of nonlinear theory of shallow water, which allows for an analytical solution using the Carrier–Greenspan transform [1]. Various examples of incident waves are considered in the literature. A review of old works is given in [2]. We also cite the latest publications [3–6]. However, in all the papers cited, the incident wave was symmetric or antisymmetric with equal steepness of the leading and trailing slopes. As a result, the formulas for the runup height can be parameterized. Parameterization includes the height and length of the incident wave as well as the distance to the coast. At the same time, numerous observations during the 2004 tsunami in the Indian Ocean indicate that the wave approaching the coast was strongly deformed with notable steepness of the leading front. We shall show that a wave with increased steepness of the leading front penetrates inland over longer distances than a wave with a symmetric profile.

It is known that the wavelength of a tsunami is sufficiently large. Therefore, the nonlinear theory of shallow water is an adequate model for the description of runup of tsunami waves on the coast. Assuming that the fluid is ideal and the wave propagates normal to the coast, let us write the main equations for the shallow water as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}([h(x) + \eta]u) = 0, \qquad (1)$$

where η is the elevation of the water surface, u is horizontal velocity of the water flow, g is acceleration due to gravity, and $h(x)$ is the unperturbed depth of the basin. Figure 1 shows the geometry of the problem: the runup zone with a length of L (the slope of the coast is constant and equal to α) continues as a flat bottom. We assume that the wave propagates to the coast from the right side and its form is specified at point $X + L$ (such a situation is usually realized during laboratory modeling of wave propagation from a wave generator).

Let us, first, consider wave motion over a flat bottom ($x > L$). In this case, the solutions to the equations of shallow water can be found in the form of a plane (Riemann) wave (see, for example, [7]):

$$\eta(x, t) = \eta_0\left(t + \frac{x - X - L}{V(\eta)}\right) \quad \text{or}$$
$$t + \frac{x - X - L}{V(\eta)} = \tau(\eta), \qquad (2)$$

where $\tau(\eta)$ is an inverse function to $\eta_0(t)$ describing the form of the wave propagating from the ocean at point

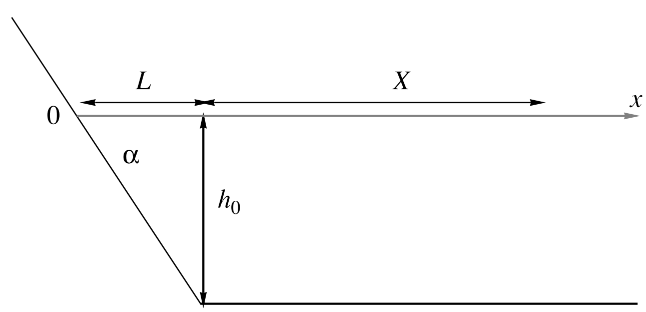

Fig. 1. Geometry of the coastal zone.

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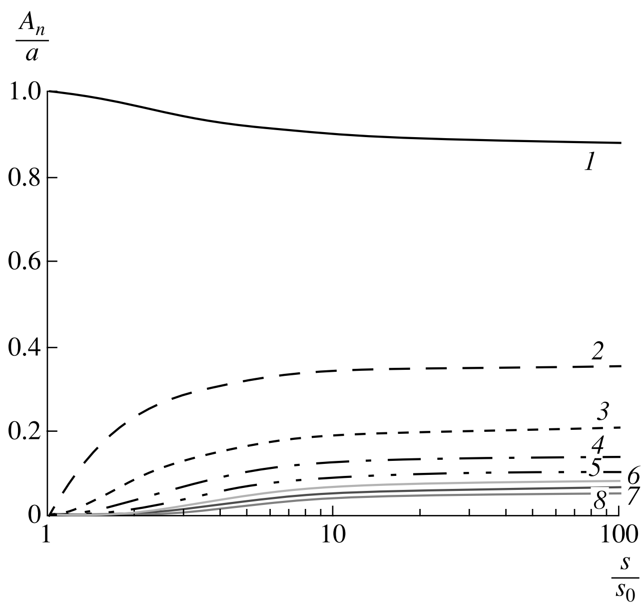

Fig. 2. Amplitude of harmonics vs. wave steepness (numerals at the curves indicate numbers of harmonics).

$X + L$ (or the wave generated by a wave producer). The characteristic velocity is

$$V(\eta) = 3\sqrt{g(h+\eta)} - 2\sqrt{gh}. \qquad (3)$$

The nonlinear wave deformation in shallow water within (2) is well known and can be analyzed for a wave of arbitrary amplitude. Nevertheless, further analysis will be based on the approximation of weak nonlinearity, when the characteristic velocity is approximated by a linear (with respect to the wave height) expression

$$V(\eta) \approx c\left(1 + \frac{3\eta}{2h_0}\right), \quad c = \sqrt{gh_0}. \qquad (4)$$

The steepness of the wave increases with distance, and its local steepness ($\left(s = \frac{\partial\eta}{\partial x}\right)$) can be found easily from (2):

$$s(x) = \frac{s_0}{1 - \dfrac{X+L-x}{L_n}}, \qquad (5)$$

where s_0 is the steepness of the incident wave approaching the coast, and L_n is the length of the nonlinearity determining the distance where the wave breaks. In particular, if a sinusoidal wave $\eta(t) = a\sin(\omega t)$ approaches the coast, the wave overturns at a distance

$$L_n = \frac{2ch_0}{3\omega a}, \qquad (6)$$

and its steepness turns to infinity $\left(\text{initial steepness is } s_0 = \frac{a\omega}{c}\right)$.

Nonlinear wave deformation in shallow water leads to the generation of higher harmonics, which can be calculated in the approximation of weak nonlinearity (problems of this kind are actively studied in nonlinear acoustics [8, 9]):

$$\eta(\tau, x) = \sum_{n=1}^{\infty} A_n(x)\sin\left[n\omega\left(t + \frac{x-X-L}{c}\right)\right], \qquad (7)$$

where the amplitudes of harmonics depend on distance

$$A_n(x) = 2a\frac{L_n}{n(X+L-x)}J_n\left(\frac{n(X+L-x)}{L_n}\right), \qquad (8)$$

where J_n is Bessel function. However, it is more convenient to exclude the distance and find a relation for the spectrum of the wave (8) with steepness (5) (this dependence is shown in Fig. 2):

$$A_n(s) = \frac{2a}{n\left(1 - \dfrac{s_0}{s}\right)}J_n\left(n\left[1 - \frac{s_0}{s}\right]\right). \qquad (9)$$

The amplitudes of harmonics increase with increasing steepness and tend to limiting values at high steepness, while the amplitude of the first harmonic decreases. As a result, it is possible to estimate the spectrum of a nonlinearly deformed wave on the basis of the measured wave steepness and actually exclude the stage of propagation of the wave over horizontal bottom from the analysis of the runup problem. A wave of type (7) with amplitudes (9) at point $x = L$ $\left(t - \frac{X}{c}\right.$ can be designated as new time $\left.t\right)$ can be considered as the initial wave in the solution to the problem of transformation and runup of the wave on a flat slope.

In the region $0 < x < L$, it is necessary to return to the solution to nonlinear equations in shallow water (1). It is obtained using the Carrier–Greenspan transform mentioned above. In particular, if we are interested only in the maximal runup height (as well as maximal drawdown depth), it is enough to consider a linear problem about fluctuations of the water level at the shoreline. The mathematically strict result is described in [2]. The solution to the linear problem is obtained quite easily, and the fluctuations of water level along the shoreline are written as

$$R(t) = \eta(0, t) = P\sum_n \sqrt{n}A_n\cos\left(n\omega t + \frac{\pi}{4}\right), \qquad (10)$$
$$P = 2\pi\sqrt{\frac{2L}{\lambda}},$$

where λ is the wavelength over the interval of the flat bottom determined from the given wave frequency ω using a linear dispersion relation $\left(\lambda = \frac{2\pi c}{\omega}\right)$. If the wave is monochromatic, then it is easy to determine from (10) that the maximal wave height is $R_{\sin} = aP$ (see, for example, [2]). The drawdown depth in a sinusoidal wave is equal to the runup height. The distances of runup and drawdown are naturally the same and equal to $X_{\sin} = \frac{R_{\sin}}{\alpha}$. When nonlinearly deformed wave (7) approaches the coast, the maximal runup height and drawdown depth (normalized by the runup height of the sinusoidal wave $R_{\sin}$) are calculated numerically using system (10). The results of calculations are shown in Fig. 3. We see that the drawdown depth only slightly depends on the wave steepness (the variation is not more than 30%), and the estimates for the drawdown based on expression derived for a monochromatic wave are satisfactory. On the contrary, the runup height strongly depends on the wave steepness and tends to infinity for a shock wave (bore) within this model (actually, the wave breaking limits the onshore wave height). In the first approximation, this curve can be shown as an almost square root dependence

$$R_{\max} = 2\pi a\sqrt{\frac{2L}{\lambda}\left[\frac{s}{ak}\right]^{0.47}}. \tag{11}$$

In the previous modeling of a tsunami wave runup, it was mentioned repeatedly that the runup height is related to the amplitude of the approaching wave by a nonlinear relation (see [2] and references therein). In particular, for a soliton, the wavelength depends on amplitude as $\lambda \sim a^{-1/2}$, which gives $R \sim a^{5/4}$. However, a nonlinear dependence is obtained for the wave of any symmetrical form, although the length is not related to its amplitude. In [10], a qualitative explanation of this effect related to nonlinear wave deformation is suggested, but no quantitative explanation and formulation of the determining parameters are given. Within the framework of the theory suggested in the present paper, the role of the wave steepness as a determining parameter for the calculation of the tsunami wave runup height becomes clear. The theory developed here demonstrates that runup of a breaking wave (or at least a strongly deformed wave) can be significantly stronger than the runup of a symmetrical wave. The observations of deep inland penetration of a wave in breaking stage (including the catastrophic tsunami in 2004 in the Indian Ocean) can also be interpreted within this theory.

If the runup height and drawdown depth are known, it is possible to calculate the distance of runup and drawdown because the angle of the coast slope is known. Nonlinear theory also makes it possible to study the characteristics of wave breaking on the coast. In particular, when the wave approaches the coast, nonlinear wave deformation leads to earlier wave breaking.

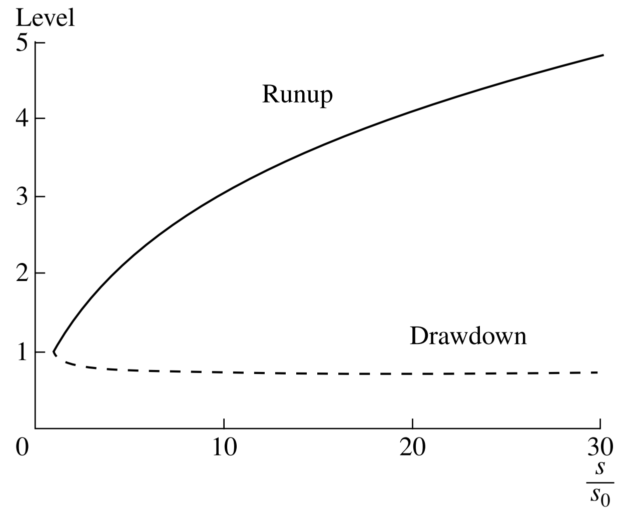

Fig. 3. Runup height and drawdown depth vs. steepness of the incident wave.

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REFERENCES

1. G. F. Carrier and H. P. Greenspan, J. Fluid Mech. **4**, 97 (1958).
2. E. N. Pelinovsky, *Hydrodynamics of Tsunami Waves* (IPF RAN, Nizhni Novgorod, 1996) [in Russian].
3. S. R. Massel and E. N. Pelinovsky, Oceanologia (Poland) **43** (1), 61 (2001)
4. G. F. Carrier, T. T. Wu, and H. Yeh, J. Fluid Mech. **475**, 79 (2002).
5. U. Kânoğlu, J. Fluid Mech. **513**, 363 (2004).
6. S. Tinti and R. Tonini, J. Fluid Mech. **535**, 33 (2005).
7. N. E. Vol'tsinger, K. A. Klevanny, and E. N. Pelinovsky, *Longwave Dynamics of the Coastal Zone* (Gidrometeoizdat, Leningrad, 1989) [in Russian].
8. O. Rudenko and S. Soluyan, *Theoretical Principles of Nonlinear Acoustics* (Nauka, Moscow, 1975) [in Russian].
9. S. N. Gurbatov, A. I. Malakhov, and A. I. Saichev, *Nonlinear Random Waves in Dispersion-Free Media* (Nauka, Moscow, 1990) [in Russian].
10. N. N. Osipenko and E. N. Pelinovsky, Okeanologiya **32**, 640 (1992).