

Risk Assessment based on the Mathematical Model of Diffuse Exogenous Geological Processes

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Abstract The estimation of the risk caused by hazardous geological processes is a common problem. The aim of this paper is to present a method for solving problems involving a wide spectrum of diffuse exogenous geological processes, based on the mathematical morphology of landscapes. Diffuse processes develop as random rounded sites within certain areas and include karstification, subsidence, thermokarstification and aeolian processes. Model assumptions take into consideration spatial and temporal randomness of occurrence, independent growth under occasional conditions, and cessation of growth given a ‘stop factor’, with transformation into degenerate sites. Based on these model assumptions, the following probabilities of impact by diffuse exogenous geological processes are solved for analytically. Firstly, the probability of impact on engineering constructions of round shape given the size of the diffuse processes and the known average impact risk; secondly, the probability of impact on small engineering constructions (points); and lastly, the probability of impact on linear engineering constructions of specified length. The computed impact probabilities are verified in various areas within Western Siberia.

Keywords Mathematical modeling · Risk assessment · Diffuse exogenous processes · Landscape pattern

Introduction

Estimation of risk to an engineering construction caused by geological phenomena is a common problem. Most research deals with the task of risk estimation (Ansell and Wharton 1992; Kolluru et al. 1996; Hu et al. 2001; Ragozin 2003). Typically, the

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following measures are used to quantify risk to an engineering construction by geological phenomena. The probability that an engineering construction will be affected to some degree by a hazardous geological process; and the distributions describing the expected value of damage to an engineering construction at various magnitudes of the hazardous process. One of the most commonly used methods of obtaining such distributions is to observe and obtain statistical data for similar engineering constructions impacted by the same hazardous geological process. However, the difficulty with this approach is in accumulating sufficient observational data. The goal of this paper is to present a risk estimation technique utilizing a mathematical model of the hazardous process and remote sensing data. This technique may be applied to a wide range of diffuse exogenous geological processes.

Diffuse exogenous processes occur at random rounded sites within areas of process development (Fig. 1). They include karstification, subsidence, thermokarstification and aeolian processes (wind erosion dimples). The term comes from the definition of diffuse natural processes used in previous work (Viktorov 1973). This paper deals with a simple case in which an engineering construction is impacted under a uniform condition. The uniform condition is characterized by relative permanency of surface deposits and the presence of geological structure without disjunctive dislocations or buried channels.

The proposed methodology is based on techniques involving the mathematical morphology of landscapes. This is a new branch of geosciences that studies quantitative laws for the patterns formed by natural units (landscapes) and develops techniques for their mathematical analysis. The basic concept behind mathematical morphology is the elaboration of a mathematical model for a landscape pattern (Victorov 1998, 2005; Victorov and Trapeznikova 2000). The landscape pattern model is a set of mathematical equations describing its most essential geometric features and can be used as the basis for solving tasks using mathematical analysis.

Theoretical Analysis

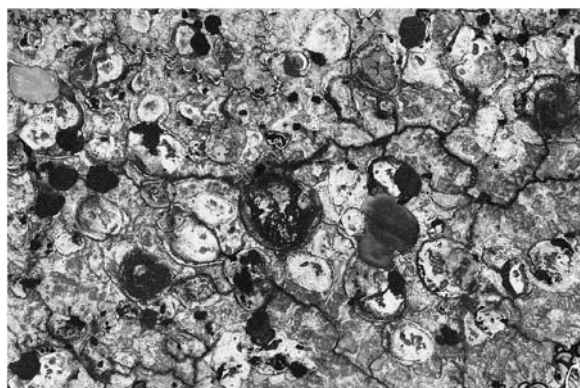
The Basis of the Model

Risk from the geometric characteristics of diffuse process development can be estimated based on the morphology of landscapes. Analysis of the process in question provides a foundation for the following model assumptions. First, the occurrence of a new hazard site during any time period and within any area is an event independent of other hazard site occurrences. The hazard site probability of occurrence is directly proportional to the time period and size of the area under study. The probability that two or more hazard sites occur is negligibly small compared to the occurrence of only one hazard site. A new hazard site cannot appear within an existing hazard site. Secondly, changes in a new site of isometric form are described as an occasional process $F_0(x, t)$ (probability density function $f_0(x, t)$) if obstacles are absent. We expect the distribution to have statistical moments of order greater than three. Different sites change independently of each other. Finally, site growth can stop after reaching a

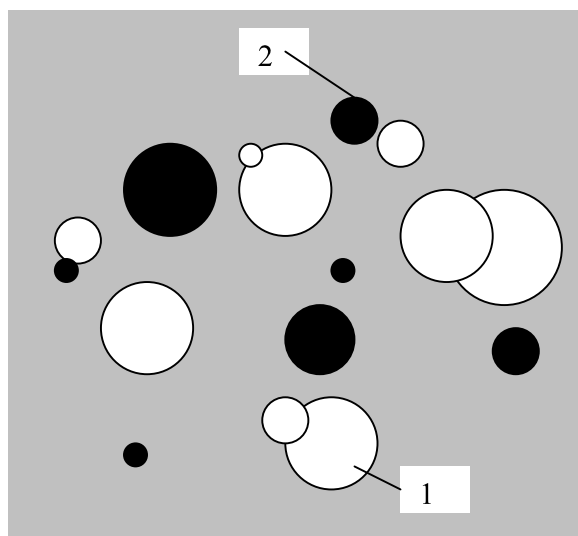
Fig. 1 a Remote sensing image of a typical morphological pattern of a plain impacted by diffuse exogenous geological processes (thermokarst), scale 1:50000, Western Siberia.

b Scheme of the morphological pattern of a territory appearing under a diffuse exogenous geological process;

1—degenerated sites of different age, 2—developing sites



(a)



(b)

critical size having distribution $F(x)$. When the site stops growing immediate “degeneration” takes place. The critical size of the site does not depend upon other sites. A scheme of an area displaying this pattern is shown in Fig. 1b.

Cessation of site growth can occur for various reasons (stop factors) depending on the geological process. For example, they may be the presence of geomechanical features within a karst layer, isolation resistance, or thermo-karst lake draining. Therefore, the region under study can be characterized by a stochastic progression of hazard sites appearing and developing independently of each other. Analogous to rain drops falling on a water surface, where every drop provides widening circles independently, the model is named the rainfall model. The model contains the main dynamical equations of the process (Victorov 2003).

The model assumptions enable us to draw conclusions regarding the ‘generation of initial sites’. The first set of assumptions results in the Poisson law for generation

of initial sites. Hence, given constant density of initial site generation, the probability of k initial sites occurring within area s during time t is described by

$$P(k, t) = \frac{(\lambda st)^k}{k!} e^{-\lambda st}, \quad (1)$$

where λ is the average number of sites appearing within a unit area per unit time.

Taking into account the second set of assumptions we find that new sites occur only outside areas already affected by the process. This set of assumptions is equivalent to the variable density of initial sites generated by

$$\lambda_1(t) = \lambda[1 - P_d(t)], \quad (2)$$

where $P_d(t)$ is the probability that a random point is within any area affected by the process. $P_d(t)$ represents the average area impacted by the process within the total site area (the area occupied within active sites). In this case, development of a number of initial sites at time t is described by

$$P(k, t) = \frac{[\gamma(t)s]^k}{k!} e^{-\gamma(t)s}, \quad (3)$$

where $\gamma(t)$ is the average number of sites per unit area at time t and is given by

$$\gamma(t) = \lambda \int_0^t [1 - P_d(u)] du.$$

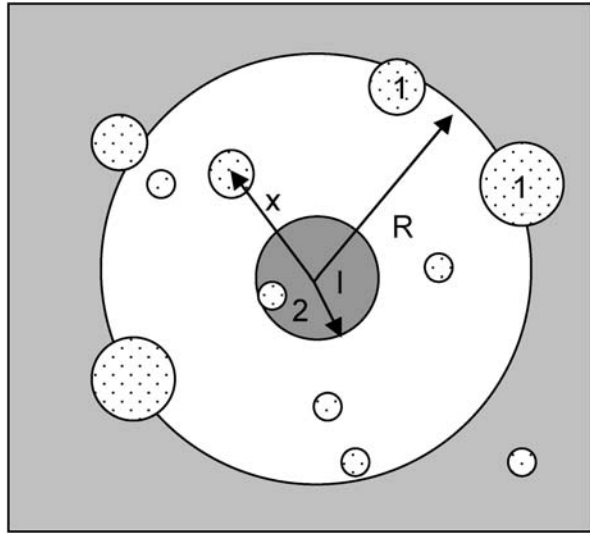
Finally, taking into account the stop-factor, the real size distribution of active sites at time t is given by

$$f_1(x, t) = \frac{[1 - F(x)] \int_0^t [1 - P_d(u)] f_0(x, t - u) du}{\int_0^{+\infty} [1 - F(x)] \int_0^t [1 - P_d(u)] f_0(x, t - u) du dx}. \quad (4)$$

Theoretical Risk Assessment for Areal (Spatial) Constructions and Small Area (Point) Constructions

Theoretical assessment of risk to an engineering construction by exogenous diffuse geological processes can be achieved on the basis of the model described above. The following analytical transform for theoretical risk assessment is valid when the age of the construction coincides with the beginning of the process. The equations can be modified for the case when they do not coincide. The model that has been developed allows us to determine the probability that an areal (spatial) construction will be affected by a diffuse geological process. Consider a construction of circular shape as an example. It is necessary to obtain the probability of impact on the round plot, of given size 1, by the geological process. Let us confine the area under consideration to a circle of radius R , whose center is placed in the middle of the round plot (Fig. 2).

Fig. 2 An illustration for the development of formula (5): l —process sites, 2—areal construction (see explanation in the text)



The probability that a hazard site, which has occurred at some distance x from the center, does not make contact with the round plot during time t is

$$\alpha = \int_l^R F_1(x-l, t) \frac{2x}{R^2 - l^2} dx, \quad (5)$$

where $F_1(x, t)$ is the distribution of radii of active sites at time t if a stop-factor exists.

Assuming independence between the arrangement and size of sites and recalling the absence of site centers within the plot, one can obtain the probability that none of the k sites originating within the area makes contact with the round plot

$$P_{R,k}(l) = \alpha^k \frac{[\gamma(t)\pi(R^2 - l^2)]^k}{k!} e^{-\pi\gamma(t)R^2}. \quad (6)$$

By summing the series over k and converting, we obtain the probability that the round plot does not make contact with a site occurring within an area of radius R

$$P_R(l) = \exp \left[2\pi\gamma(t) \int_l^R x F_1(x-l, t) dx - \pi\gamma(t)R^2 \right]. \quad (7)$$

After simplification, we obtain

$$P_R(l) = \exp \left\{ -2\pi\gamma(t) \int_l^R x [1 - F_1(x-l, t)] dx - \pi\gamma(t)l^2 \right\}. \quad (8)$$

The probability that the round plot is not affected by any site undergoing the geological process is obtained by evaluating the limit of (8), as R goes to infinity

$$P(l) = \lim_{R \rightarrow +\infty} P_R(l). \quad (9)$$

The task of deriving analytically the probability $P(l)$ is reduced to evaluating the integral

$$I = 2 \int_l^{+\infty} x [1 - F_1(x - l, t)] dx. \quad (10)$$

A change of variables and partial integration results in

$$I = \int_0^{+\infty} (v^2 + 2vl) f_1(v, t) dv + \lim_{v \rightarrow +\infty} (v^2 + 2vl) [1 - F_1(v, t)], \quad (11)$$

where $f_1(x, t)$ is the density distribution of active sites at time t . Following a change of variables and the use of L'Hopital's rule, we find the limit to equal 0 due to the existence of the second and third moments for the distribution $F_1(x, t)$. This means that the integral converges and that the desired probability is

$$P(l) = e^{-\pi\gamma(t)[(\bar{r}(t)+l)^2 + \sigma_r^2(t)]}. \quad (12)$$

Correspondingly, the probability of impact on a round plot of radius l is

$$P_{ds}(l) = 1 - e^{-\pi\gamma(t)[(\bar{r}(t)+l)^2 + \sigma^2(t)]}, \quad (13)$$

where $\gamma(t)$ is the average number of active sites per unit area at time t , $\bar{r}(t)$ is the average radius of active sites at time t , and $\sigma(t)$ is the standard deviation of the radii at time t . For estimating probability during period $[0, t]$, it is necessary to take into consideration not only the active sites at time t , but also those sites that were active earlier but became degenerate prior to time t . Equation (13) also applies for degenerate sites. Assuming independence of active and degenerate sites, the resulting formula is

$$P_{ds}(l) = 1 - e^{-\pi\gamma(t)[(\bar{r}(t)+l)^2 + \sigma^2(t)] - \pi\gamma_{dg}(t)[(\bar{r}_{dg}(t)+l)^2 + \sigma_{dg}^2(t)]}, \quad (14)$$

where $\gamma_{dg}(t)$, $\bar{r}_{dg}(t)$, $\sigma_{dg}(t)$ are the average number of degenerate sites per unit area, the average radius of degenerate sites, and the standard deviation of the radii of degenerate sites at time t , respectively. The described mathematical model provides a solution for the probability that small areal constructions will be affected by a diffuse geological process. The small construction should be regarded as a point construction. From the interpretation given above, this probability is in fact the required probability of impact. The average impact is also a limit probability for an areal construction. Taking the limit of (13), as $l \rightarrow 0$, and simplifying results

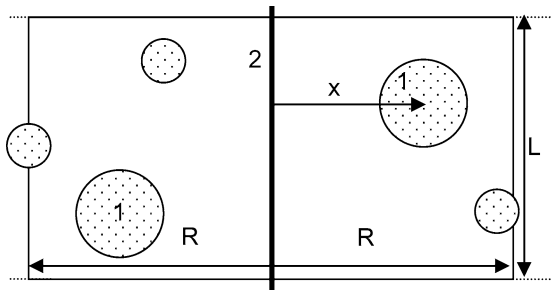
$$P_d(t) = 1 - e^{-\gamma(t)s(t)}, \quad (15)$$

where $s(t)$ is the average area of active sites at the given moment. Taking into consideration the probability of impact when degenerate sites are present, (15) becomes

$$P_{ds}(t) = 1 - e^{-\gamma(t)s(t) - \gamma_{dg}(t)s_{dg}(t)}, \quad (16)$$

where $s_{dg}(t)$ is the average area of the degenerate sites at time t .

Fig. 3 An illustration for the development of formula for impact on a linear construction: 1—process sites, 2—linear construction (see explanation in the text)



The mathematical expectation of the impact value of geological hazards for a circular plot of radius R is an important parameter of risk assessment for spatial constructions. The impact value for the construction units within a large area (compared to the site area), S , can be obtained by considering the probability of impact at any given point of the construction, as described by (16). Correspondently, the mathematical expectation for the impact value is

$$R_s(S) = [1 - e^{-\gamma(t)s(t) - \gamma_{dg}(t)s_{dg}(t)}]S. \quad (17)$$

Theoretical Risk Assessment for Linear Constructions

Consider a linear construction of length L crossing a uniform section affected by the diffuse geological process in question. Let the section be an infinite band of corresponding width, L , perpendicular to the linear construction (Fig. 3). A number of intersections of the linear construction for active sites of the diffuse process need to be analyzed. Consider a band of a finite length $2R$. The probability that at least one active site makes contact with the linear construction depends upon its radius and the distance between its center and the construction

$$\alpha = \int_0^R [1 - F_1(x, t)] \frac{dx}{R}. \quad (18)$$

The probability that v sites make contact with the linear construction, if the total number of sites within the band is k , is

$$P(v, k, R) = \binom{k}{v} \alpha^v (1 - \alpha)^{k-v} \frac{(2\gamma(t)RL)^k}{k!} e^{-2\gamma(t)RL}. \quad (19)$$

Correspondingly, the number of contacts given a random number of sites within the band can be obtained by summing over k

$$P_R(v) = \frac{(2\alpha\gamma(t)LR)^v}{v!} e^{-2\alpha\gamma(t)LR}. \quad (20)$$

The expression for the infinite band is obtained by taking the limit of (20) as $R \rightarrow +\infty$. When

$$\lim_{R \rightarrow +\infty} \alpha R = \int_0^{+\infty} [1 - F_1(x, t)] dx = \lim_{R \rightarrow +\infty} x[1 - F_1(x, t)] + \int_0^{+\infty} x f_1(x, t) dx \quad (21)$$

we obtain

$$\lim_{R \rightarrow +\infty} \alpha R = \bar{r}(t) \quad \text{and} \quad P(v) = \frac{[2\gamma(t)L\bar{r}(t)]^v}{v!} e^{-2\gamma(t)L\bar{r}(t)}, \quad (22)$$

where $\bar{r}(t)$ is the average radius. The resulting distribution of the number of impacts on a linear construction from hazard sites of the diffuse process is a Poisson distribution with a free parameter. The parameter is equal to the product of the length of the linear construction, the site distribution density and their average diameter. The impact probability for linear constructions is

$$P_{dl}(L) = 1 - e^{-2\gamma(t)\bar{r}(t)L}. \quad (23)$$

Including the possible impact on the construction by degenerate sites, which were active prior to the current time t , we obtain (for linear constructions)

$$P_{dl}(L) = 1 - e^{-2[\gamma(t)\bar{r}(t) + \gamma_{dg}(t)\bar{r}_{dg}(t)]L}, \quad (24)$$

where $\gamma(t)$ is the average number of sites per unit area and $\bar{r}(t)$ is the radius of active sites at time t , and $\gamma_{dg}(t)$ and $\bar{r}_{dg}(t)$ are the same parameters for degenerate sites.

Let us find the mean impact of the geological hazard on a linear construction of length L . The probability that the length of a segment within an active site is greater than x is

$$P(\xi > x) = \int_{\frac{x}{2}}^{+\infty} \frac{1}{v} \sqrt{v^2 - \left(\frac{x}{2}\right)^2} f_1(v, t) dv. \quad (25)$$

By converting, we obtain the probability distribution of one site of the diffuse process impacting a linear construction to be

$$F_{rl}(x, t) = 1 - \frac{x}{2} \int_1^{+\infty} \sqrt{1 - \frac{1}{u^2}} f_1\left(u \frac{x}{2}, t\right) du. \quad (26)$$

Assuming the moment of second order is finite,

$$\int_0^{+\infty} x f(x) dx = \int_0^{+\infty} [1 - F(x)] dx, \quad (27)$$

and we obtain

$$\bar{r}_l(t) = \int_0^{+\infty} \frac{x}{2} \int_1^{+\infty} \sqrt{1 - \frac{1}{u^2}} f_1\left(u \frac{x}{2}, t\right) du dx. \quad (28)$$

By converting, we obtain

$$\bar{r}_l(t) = \int_1^{+\infty} \frac{2}{u^2} \sqrt{1 - \frac{1}{u^2}} \bar{r}(t) du. \quad (29)$$

This means that the mean impact of geological hazards on one construction is

$$\bar{r}_l(t) = \bar{r}(t) \frac{\pi}{2}. \quad (30)$$

From (16), which calculates the probability of impact on a point of the linear construction, we obtain the average impact

$$R_l(L) = [1 - e^{-\gamma(t)s(t) - \gamma_{dg}(t)s_{dg}(t)}]L. \quad (31)$$

Empirical Issues

Testing

The resulting equations were tested in a number of regions. The impact probability formula for point constructions was tested in a region of northwestern Siberia, where thermo-karst processes have occurred (Fig. 1a). The terrain in the area is sub-horizontal and weakly undulating with vegetation typical of the tundra. Numerous lakes and khasyreis (drained thermo-karst lakes) are present. A sparse erosion net has also developed in various locations. The lakes are characterized by isometric, often circular form and are randomly scattered on the plain. The khasyreis appear to be flat depressions of similar isometric form. They are covered with meadow or bog vegetation and are also randomly scattered on the plain. Thermo-karst, thermo-abrasion, and thermo-erosion processes have a complex relationship and influence this type of area. New thermo-karst depressions are filled with water, become lakes, and increase in size independently of each other due to the thermo-abrasion process. Their growth depends upon random factors related to yearly meteorological processes. Occasionally a lake can be drained as a result of fluvial erosional processes and becomes a khasyreis, a lacustrine depression containing meadow and bog vegetation. In turn, the khasyreis becomes an area of developing permafrost, where new initial thermo-karst depressions can occur. This environment agrees with the definition of diffuse exogenous processes and can be described by the model elaborated in this paper. Thermo-karst lakes are active sites of thermo-karstification while khasyreis are degenerate sites. Lake drainage due to the fluvial erosional process corresponds to the stop factor.

For testing purposes, we first sample locations for construction by using a random number generator to obtain point coordinates. The points are then marked on an aerial photograph of the area affected by the thermo-karst process. Then, the empirical frequency of impacts is calculated as the number of points located within impacted sites, both active (thermo-karst lakes) and degenerate (khasyreis). The theoretical impact probability at a point is calculated for the same area using the formulae given in the previous section. Both lakes and khasyreis are taken into consideration, as any degenerate site was once an active one. The theoretical probability is given by

$$P_d(t) = 1 - e^{-\gamma_l(t)s_l(t) - \gamma_h(t)s_h(t)}, \quad (32)$$

where $s_l(t)$ and $\gamma_l(t)$ are the average area and the average number of thermo-karst lakes per unit area at time t , respectively. $s_h(t)$ and $\gamma_h(t)$ are the same parameters for the khasyreis.

Similar tests, appropriately modified, were carried out in an area in Middle Siberia, where ancient thermo-karsts developed, and in an area in the Caspian lowland, where

Table 1 Impact probabilities for point constructions

Statistical parameter	Value
Thermo-karst process, North Western Siberia	
Theoretical impact probability for a point unit	0.451
Empirical frequency of impact on a point unit	0.440
Number of samples	100
Acceptable deviation at 0,95	0.10
Soil subsidence, the Caspian lowland	
Theoretical impact probability for a point unit	0.204
Empirical frequency of impact on a point unit	0.208
Number of samples	240
Acceptable deviation at 0,95	0.066
Ancient thermo-karst, Middle Siberia	
Theoretical impact probability for a point unit	0.260
Empirical frequency of impact on a point unit	0.343
Number of samples	100
Acceptable deviation at 0,95	0.10

subsidence processes have taken place. The Middle Siberian area is a developed arable territory containing remains of ancient thermo-karsts which have resulted in randomly scattered isometric depressions. The area in the Caspian lowland has flat topography and is covered with some vegetation. The flat terrain is perturbed by round depressions representing soil subsidence. In contrast to the previous example, regular grids of point locations were used in the modeling. Theoretical impact probabilities were estimated using (15), and theoretical and empirical data was compared in Table 1. The empirical data substantiates the proposed model.

The impact probability formula for a round plot of given size was tested in the same two areas. In order to verify equations describing the impact probability for a round engineering construction, circles simulating engineering constructions were located within the area on a uniform grid. The impact frequency was calculated. Average density, radius, and standard deviations of subsidence depressions required for (13), and used in estimating impact probability for round constructions of various sizes, were determined using aerial images and maps. Table 2 compares the theoretical and empirical probability values for constructions of various sizes for the area in Middle Siberia occupied by ancient thermo-karst depressions. With acceptable deviation of 0.100 at 0.95 confidence, the empirical probabilities agree well with the calculated probabilities. Similar results were obtained in comparing theoretical and empirical probabilities for the area in the Caspian lowland where soil subsidence depressions are present (Table 3). Analysis of the table shows that the calculated impact probability for a round plot is insignificantly different from the empirical probability.

Table 2 Impact probabilities for round constructions of varying size in an area where ancient thermo-karstification has occurred (Middle Siberia)

Size of construction	Theoretical probability	Empirical frequency	Difference
10	0,367	0,393	0,026
20	0,477	0,505	0,028
30	0,582	0,639	0,057
40	0,677	0,75	0,073
50	0,76	0,835	0,075
60	0,826	0,882	0,056
70	0,878	0,916	0,038

Table 3 Impact probabilities for round plots of varying size in an area where soil subsidence has occurred (Caspian lowland)

Size of a plot	Theoretical probability	Empirical frequency	Difference	Acceptable deviation at 0.95
2	0.303	0.356	0.053	0.067
4	0.410	0.458	0.048	0.065
6	0.518	0.593	0.075	0.099
8	0.620	0.759	0.139	0.107

Practical Application

The described model can be used in practical applications for determining the probability of impact on an engineering construction and in evaluating possible damage. Consider an example of a linear construction, such as a pipeline. The probability that thermo-karstification will impact a straight linear construction of length (L) within a uniform area of a plain undergoing thermo-karstification and fluvial erosional processes, during operating time T , is considered. The task can be solved using the mathematical model for the morphological pattern of thermo-karst plains with fluvial erosion (Victorov 2005). We assume that the thermo-karsts developed well before the pipeline was constructed. The impact probability depends on the two probabilities p_1 and p_2

$$p = 1 - (1 - p_1)(1 - p_2), \quad (33)$$

where p_1 is the impact probability on an engineering construction due to growth of thermo-karst sites that appeared before the construction was built and p_2 takes into account the impact by new sites after the construction was built. The first probability assumes independence of thermo-karst site development from development of other sites and is given by the product

$$p_1 = 1 - \prod_{i=1}^n [1 - q_i(t)], \quad (34)$$

where

$$q_i(t) = \frac{1 - F(r_i)}{1 - F(v_i)} \int_{r_i}^{+\infty} f^0(u, v_i, t) du.$$

r_i is the distance between the center of the i -th existing thermo-karst site and the linear construction. v_i is the radius of thermo-karst site i . $f^0(x, v, t)$ is the probability that the site expands to radius x after time t given initial radius v . $F(x)$ is the distribution function for the critical size and n is the number of active thermo-karst sites at the time the pipeline was built. The first multiplier $q_i(t)$ determines the probability that a site will continue to grow until it affects the pipeline without ceasing due to a stop-factor. Hence, $f^0(x, v, t)$ is the transition probability function when the diffuse process is described by a Markov process. The required function for the morphological pattern of thermo-karst plains with fluvial erosion (Victorov 2005) is given by the equations

$$f^0(x, v, t) = \frac{1}{\sqrt{2\pi}\sigma x\sqrt{t}} e^{-\frac{(\ln \frac{x}{v} - at)^2}{2\sigma^2 t}}, \quad (35)$$

$$F(x) = 1 - \exp(-\pi\gamma x^2), \quad (36)$$

where a is the growth rate of a mean logarithm for the ratio between finite and initial dimensions of thermo-karst depressions, σ^2 is the growth rate of the standard deviation logarithm for the ratio between finite and initial dimension of thermo-karst depressions, and γ is the average density of fluvial erosion sources. The expression for $q_i(t)$ is given by the equation

$$q_i(t) = e^{-\pi\gamma(r_i^2 - v_i^2)} \int_{r_i}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma u\sqrt{t}} e^{-\frac{(\ln \frac{u}{v_i} - at)^2}{2\sigma^2 t}} du. \quad (37)$$

The second probability p_2 is obtained from a series of assumptions similar to those listed for (18) to (23). If a site appears at distance x from the linear construction (Fig. 3), at time t after the period $[0, T]$, the impact probability for a linear construction is determined by the probabilities that the critical size (stop factor) and the radius of the site are larger than the distance from the site center to the linear construction

$$\alpha = \int_0^T \int_0^R [1 - F(x)][1 - F_0(x, t)] \frac{dx dt}{R T}, \quad (38)$$

where $F_0(x, t)$ is the radius distribution function for thermo-karst sites on a thermo-karst plain with fluvial erosion. According to Victorov (2005), $F_0(x, t)$ follows the lognormal distribution

$$F_0(x, t) = \int_0^x \frac{1}{\sqrt{2\pi}\sigma u\sqrt{t}} e^{-\frac{(\ln u - at)^2}{2\sigma^2 t}} du. \quad (39)$$

The probability that v sites make contact with the linear construction, if the total number of sites within the band is k , is

$$P(v, k, R, T) = \binom{k}{v} \alpha^v (1 - \alpha)^{k-v} \frac{[2\lambda(1 - P)TRL]^k}{k!} e^{-2\lambda(1-P)TRL}, \quad (40)$$

where λ is the average number of sites occurring within a unit area per unit time and P is the area outside active thermo-karst sites. Equation (40) assumes that new thermo-karst sites do not appear within already existing active sites. After prolonged thermo-karst development, P becomes constant under rather general conditions (Victorov 2003). After summing over k and taking the limit as in (20)–(23), we obtain the expression for the impact probability on an engineering construction

$$p_2 = 1 - e^{-2\beta(T)\lambda(1-P)L}, \quad (41)$$

where $\beta(T)$ is determined by the equation

$$\beta(T) = \lim_{R \rightarrow +\infty} \alpha RT. \quad (42)$$

The analytical formula for (42) is obtained by using (38)

$$\beta(T) = \int_0^T \int_0^{+\infty} [1 - F(x)][1 - F_0(x, t)] dx dt. \quad (43)$$

This formula can be defined explicitly using the radius distribution function for thermo-karst sites (cf. (39)) and the distribution function for the critical size (cf. (36))

$$\beta(T) = \int_0^{+\infty} \exp(-\pi\gamma x^2) \int_0^T \left[1 - \Phi\left(\frac{\ln x - at}{\sigma\sqrt{t}}\right) \right] dt dx, \quad (44)$$

where $\Phi(t)$ is the Laplace function. This equation simplifies to

$$\beta(T) = \frac{T}{2\sqrt{\gamma}} - \int_0^{+\infty} \exp(-\pi\gamma x^2) \int_0^T \Phi\left(\frac{\ln x - at}{\sigma\sqrt{t}}\right) dt dx. \quad (45)$$

It is evident that the impact probability for linear constructions is determined by the parameters a , σ , λ , P , γ . All these parameters can be determined rather precisely from repeat airborne or satellite precision photos. The necessary parameters are obtained from the equations

$$\begin{aligned} a &= \frac{M_2 - M_1}{t_2 - t_1}, & \sigma &= \sqrt{\frac{D_2 - D_1}{t_2 - t_1}}, & \lambda &= \frac{(n_2 - n_1) + (m_2 - m_1)}{(t_2 - t_1)(S - S_t)}, \\ P &= \frac{S_t}{S}, & \gamma &= \frac{\varepsilon}{S}, \end{aligned} \quad (46)$$

where t_1 and t_2 are the times of the first and second surveys. M_i is the logarithm of the average radius of the thermo-karst sites (lakes) within the testing area at the

corresponding date of the survey. D_i is the variance of the logarithms of the radii of thermo-karst sites (lakes) within the testing area at the corresponding date of survey. n_i and m_i are the number of active thermo-karst sites (lakes) and degenerate sites (khasyreis), respectively, within the testing area at the corresponding date of the survey. S is the total testing area. S_i is the part of the testing area occupied by thermo-karst sites and ε is the number of fluvial sources within the testing area.

Conclusions

The mathematical model of diffuse exogenous geological processes provides a quantitative assessment of natural hazard risk with a theoretical analytical method based on the mathematical morphology of landscapes. It allows risk assessments for different types of engineering constructions, such as point, circular and linear constructions. Empirical testing verifies the risk assessment based on the mathematical model of diffuse exogenous geological processes.

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