GEOPHYSICS, VOL. 68, NO. 6 (NOVEMBER-DECEMBER 2003); P. 2055–2062, 4 FIGS., 2 TABLES.
10.1190/1.1635059

Effective media: A forward modeling view

Vladimir Grechka*

ABSTRACT

The existing effective media theories, such as Backus averaging, can be only used in media that possess certain characteristics regarding to concentration, shape, or geometry of their heterogeneities. These limitations originate from necessity of having analytical description of complex stress and strain fields that normally arise in microheterogeneous solids. The need for explicit solutions can be eliminated by computing the stresses and strains numerically. As a result, effective media can be in principle constructed for solids of arbitrary complexity.

This simple idea is tested on two 2D models, where conventional analytical effective media theories are likely to break down. The first model is an isotropic layered solid with cracks that intersect the layer interfaces, the second is a layered medium containing random inclusions. In both cases, some differences are observed between the effective stiffness coefficients obtained numerically and those derived using the existing effective media theories. For instance, it is demonstrated that Backus averaging (improperly) applied to horizontal isotropic layers with random inclusions leads to biases in the calculated vertical velocities.

Overall, the proposed technique enables us to establish the limits of applicability of conventional effective media theories. It also makes it possible to obtain quantitative estimates of the errors incurred because of violating certain assumptions of a given theory.

INTRODUCTION

Any measurements made in exploration seismology pertain to some effective quantities that average local elastic properties of the earth over a finite volume. We usually recognize at least two types of heterogeneities that are inevitably smoothed out in the process of making and interpreting the measurements. The smaller ones, which exist on atomic and molecular scales, are never of any concern. Their presence is eliminated by accepting the continuum hypothesis which, for instance, allows us to talk about a material point, the entity that has zero volume, without actually going into details of the material structure and behavior at subatomic level. In contrast, larger heterogeneities, such as grains, pores, or fractures, which are still much smaller than seismic resolution and a typical volume of interest, are described by continuum mechanics and represent the subject of the so-called effective media theories. Those theories aim at predicting overall elastic properties of a volume that has the size comparable with the wavelength used in a particular application.

The significance of effective media theories for oil and gas exploration is well recognized. For a heterogeneous rock that has a given type of microstructure, they allow one to obtain the effective (often homogeneous and anisotropic) solid which is indistinguishable from the initial one as long as propagation of low frequency waves is considered. Results of this kind are known for media with ellipsoidal inclusions (Eshelby, 1957), for finely layered structures (Rytov, 1956; Backus, 1962; Helbig, 1984; Schoenberg and Muir, 1989), and for fractured rocks (Hudson, 1980; Schoenberg, 1980; Schoenberg and Douma, 1988). Once effective media are constructed, one can attempt to invert various seismic signatures for certain microstructural parameters that were used in those effective models. The usefulness of this approach for fractured media was demonstrated by Bakulin et al. (2000a, b, c).

All existing effective media theories are applicable to either specific geometries (e.g., plane thin layers) or heterogeneities that possess certain characteristics (e.g., ellipsoidal shape or dilute concentration). The reason for adopting those simplifying and restrictive assumptions is clear: they result in such a state of deformation of an original microheterogeneous solid that the corresponding effective model can be derived analytically or semi-analytically. While it is known that any violation of assumptions used in a particular theory leads to an erroneous effective medium, none of the theories has a built-in mechanism for estimating those errors. For example, it is not obvious how to modify the results of Rytov (1956) or Backus (1962) to handle lateral heterogeneities inside thin layers. To overcome the apparent deficiencies of the existing effective media theories, I propose to compute the stresses and strains numerically in a given arbitrarily heterogeneous solid. Then, the effective

Manuscript received by the Editor January 8, 2003; revised manuscript received June 30, 2003.
*Shell International Exploration and Production, 3737 Bellaire Blvd., Houston, TX 77025. E-mail: vladimir.grechka@shell.com.

elastic constants are calculated using a straightforward volume average of the obtained (usually highly variable) stresses and strains. In fact, this simple idea has been around for quite a while. For instance, Hudson (1991) pointed out that "there is nothing in principle to prevent the use of a numerical solution of an inclusion of arbitrary complex shape." It is somewhat surprising that (to the best of my knowledge) this idea has not been implemented so far. In contrast to analytical methods, the computational approach described here yields effective media on a one-by-one basis and, therefore, does not provide immediate guidance on extending numerical results to all models that belong to a certain class. However, it appreciably broadens the types of microgeometries for which effective media can be constructed.

Once we realize that the fields of stresses and strains can be computed in a heterogeneous medium for any frequency of propagating waves, we become well positioned to tackle a number of problems that proved to be unresolvable by the existing analytical methods. While, in general, the frequency dependent behavior of any solid containing nondilute concentration of arbitrary inclusions can be studied, I restrict myself in this paper to two specific problems which I believe are relevant for exploration geophysics. First, I discuss effective media that correspond to fractured, isotropic, finely layered solids. I find that a combination of Backus (1962) averaging and the linear-slip theory (Schoenberg, 1980) used to describe the fractures leads to an incorrect effective model because these theories do not account for interaction between the fractures and the layers. Second, I examine isotropic horizontally layered media with random inclusions. I show that if one chooses to use a Backus (1962) average to compute effective model for such a medium, the vertical velocities of P- and S-waves come out underestimated.

DEFINITION OF EFFECTIVE MEDIA

I begin with the equation of motion,

$$\frac{\partial}{\partial x_j} c_{ijkl}(\mathbf{x}) \frac{\partial u_k}{\partial x_l} + \rho(\mathbf{x})\omega^2 u_i = 0, \quad (i = 1, 2, 3), \tag{1}$$

that describes the displacement vector $\mathbf{u}(\mathbf{x}, \omega) \equiv u_k(x_l, \omega)$ in the frequency (ω) domain. In this equation, the elastic stiffness tensor $\mathbf{c}(\mathbf{x}) \equiv c_{ijkl}(\mathbf{x})$ and the density $\rho(\mathbf{x})$ represent the properties of heterogeneous anisotropic solid at points $\mathbf{x} \equiv x_j \in V$ that occupy the volume V. Hereafter, I assume summation from 1 to 3 with respect to all repeating indexes.

The local stress-strain relationship in the solid under consideration is given by the conventional linear Hooke's law:

$$\tau_{ij}(\mathbf{x}) = c_{ijkl}(\mathbf{x}) \frac{\partial u_k(\mathbf{x})}{\partial x_l} \equiv c_{ijkl}(\mathbf{x}) \varepsilon_{kl}(\mathbf{x}), \quad (i, j = 1, 2, 3) \tag{2}$$

or

$$\boldsymbol{\tau}(\mathbf{x}) = \mathbf{c}(\mathbf{x})\, \varepsilon(\mathbf{x}), \tag{3}$$

where $\boldsymbol{\tau} \equiv \tau_{ij}$ and $\varepsilon \equiv \varepsilon_{kl}$ denote the stress and strain tensors, respectively. Since the stress and strain fields depend on frequency of a propagating wave, equation (3) could have been written in a more explicit form $\boldsymbol{\tau}(\mathbf{x}, \omega) = \mathbf{c}(\mathbf{x})\varepsilon(\mathbf{x}, \omega)$. In this paper, however, I will not discuss the frequency dependence of stresses and strains and concentrate exclusively on static deformations ($\omega = 0$).

The goal of effective media theories is to describe the overall elastic behavior of the volume V, which is called the representative one if its average properties characterize the medium on the scale of interest. To this end, these theories replace locally varying tensors $\boldsymbol{\tau}(\mathbf{x})$ and $\varepsilon(\mathbf{x})$ with their volume averages (e.g., Christensen, 1979; Hudson, 1991)

$$\langle \boldsymbol{\tau} \rangle = \frac{1}{V} \int_V \boldsymbol{\tau}(\mathbf{x})\, d\mathbf{x} \tag{4}$$

and

$$\langle \varepsilon \rangle = \frac{1}{V} \int_V \varepsilon(\mathbf{x})\, d\mathbf{x}. \tag{5}$$

Then, the stiffness tensor $\hat{\mathbf{c}}$ of a homogeneous effective medium is defined through the relation

$$\langle \boldsymbol{\tau} \rangle = \hat{\mathbf{c}} \langle \varepsilon \rangle. \tag{6}$$

The effective density $\hat{\rho}$ is a symmetric tensor for nonzero frequencies (Helbig, 1984); otherwise and in the context of this paper, it is given by

$$\hat{\rho} = \langle \rho \rangle = \frac{1}{V} \int_V \rho(\mathbf{x})\, d\mathbf{x}. \tag{7}$$

Equations (1)–(6) suggest the following straightforward procedure for obtaining the tensor $\hat{\mathbf{c}}$. Once elastic parameters of the representative volume V are specified, I solve the equations of motion (1) numerically. In fact, I need to find several solutions for different boundary conditions that produce a set of the stress and strain tensors, $\boldsymbol{\tau}^{(N)}(\mathbf{x})$ and $\varepsilon^{(N)}(\mathbf{x})$, respectively. Next, I treat elements of the averaged tensors $\langle \boldsymbol{\tau}^{(N)} \rangle$ and $\langle \varepsilon^{(N)} \rangle$ as coefficients of the linear system (6) which is used to determine the effective stiffnesses $\hat{\mathbf{c}}$. A minimum number of boundary conditions required for obtaining all 21 effective stiffness coefficients can be established based on the fact that tensor $\hat{\mathbf{c}}$ has equivalent representation by a real, symmetric, positive definite 6×6 matrix that has six mutually orthogonal eigenvectors in the strain-stress space (Dellinger et al., 1998). Therefore, at least six independent boundary conditions are needed to span the eigenspace of $\hat{\mathbf{c}}$. For instance, six tractions $T_{ij} = \tau_{ik} n_k^{(j)} (i, j = 1, 2, 3; i \leq j)$ applied to sides of volume V that has a shape of a parallelepiped with outer normals $\mathbf{n}^{(j)}$ allow one to compute the complete tensor $\hat{\mathbf{c}}$. Needless to say that the classical results of effective media theories obtained by Eshelby (1957), Backus (1962), Schoenberg (1980), and Schoenberg and Muir (1989) are derivable from definition (6) if a proper solution of equations (1) is available.

MODELING OF EFFECTIVE MEDIA

Here, following the general procedure outlined in the previous section, I describe details of actual computations that lead to the results discussed below.

1) I restrict myself to 2D isotropic media whose Lamé constants $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are the functions of $x_1 \equiv x$ and $x_3 \equiv z$ coordinates.
2) I solve the static (i.e., frequency $\omega = 0$) equations of motion (1) using finite-element method (implemented in

Matlab) in the $[x_1, x_3] \equiv [x, z]$ plane. This allows me to calculate the fields of stress and strain components $\tau_{xx}(x, z)$, $\tau_{xz}(x, z)$, $\tau_{zz}(x, z)$, and $\varepsilon_{xx}(x, z)$, $\varepsilon_{xz}(x, z)$, $\varepsilon_{zz}(x, z)$. Below, the notation $\tau_{ab}(x, z)$ and $\varepsilon_{ab}(x, z)$ with letters $(a, b) = (x, z)$ is used to represent the results of computations, whereas $T_{ij} = \tau_{i1} n_1^{(j)} + \tau_{i3} n_3^{(j)}$ with the numbers $(i, j) = (1, 3)$ is reserved for the boundary conditions at faces of volume V with outer normals $\mathbf{n}^{(j)}$.

3) By choosing my heterogeneous models to have the shape of the unit square (i.e., $V = [0, 1] \times [0, 1]$ in the $[x, z]$-plane) and specifying three pairs of tractions T_{11}, T_{13}, and T_{33} at its sides as boundary conditions, I construct nine equations (6) and solve the obtained overdetermined system for six effective stiffness coefficients $\hat{c}_{11}$, $\hat{c}_{13}$, $\hat{c}_{15}$, $\hat{c}_{33}$, $\hat{c}_{35}$, and $\hat{c}_{55}$ (the other stiffnesses are not constrained in 2D models). Although this means that, in principle, I can treat effective monoclinic media with the symmetry plane $[x, z]$, the lowest symmetry in our examples below is orthorhombic. Therefore, the condition

$$\hat{c}_{15} = \hat{c}_{35} = 0 \tag{8}$$

can be used to check accuracy of our computations.

4) The problem of static deformation of a heterogeneous solid involves one dimensionless quantity (the strain), as well as the quantities that have dimensions of length (e.g., the size of volume V and the displacement vector) and stress (the stress and stiffness tensors). Choosing the unit representative volume above, allows me to deal with the relative lengthes. Similarly, I operate with the relative stresses by selecting the unit tractions as the boundary conditions and measuring the local stiffnesses with respect to them. Thus, I end up by describing the deformation process in terms of the relative lengths and relative stresses.

The accuracy of computations below was tested for the following known effective media: VTI (transversely isotropic with a vertical symmetry axis) models due to horizontal fine layering (Backus, 1962), HTI (transversely isotropic with a horizontal symmetry axis) models due to the presence of vertical rotationally-invariant penny-shaped fractures in a homogeneous isotropic host rock (Schoenberg and Sayers, 1995; Bakulin et al., 2000a), and isotropic models corresponding to heterogeneous solids, where $\lambda(x, z)$ varies arbitrarily and $\mu(x, z) = \text{constant}$ (Hill, 1963). The size of finite elements and other computational parameters were selected in such a way that the maximum error in the effective $\hat{c}_{11}$, $\hat{c}_{13}$, $\hat{c}_{33}$, and $\hat{c}_{55}$ obtained in all those tests was less than 0.25%. I use this value as an estimate of accuracy of the effective stiffnesses for other models for which exact solutions are unknown.

FRACTURED FINELY LAYERED SOLID

A naive experiment

Here, I discuss a simple experiment that demonstrates one inherent limitation of the classical effective media theories. I begin with two homogeneous isotropic solids characterized by pairs of Lamé constants λ_1, μ_1 and λ_2, μ_2. Then, I cut plane layers out of the solids, mix them in concentrations ν_1 and ν_2 (clearly, $\nu_1 + \nu_2 = 1$), and glue the layers together to create a medium that varies in the z-direction. This makes an effective VTI solid (e.g., Backus, 1962). Next, I add rotationally invariant fractures that have the excess normal and tangential compliances K_N and K_T, respectively. The fracture normals are oriented along the x-axis. This results in an effective orthorhombic medium (Schoenberg and Helbig, 1997; Bakulin et al., 2000b). I denote it ORT1.

Exactly the same medium can also be manufactured by interchanging the procedures of mixing and fracturing the layers. Specifically, I start with putting the same fractures (characterized by the excess compliances K_N and K_T) into the original isotropic media and get two HTI solids (Schoenberg and Sayers, 1995; Bakulin et al., 2000a). Next, I cut these solids into the same layers as before and mix them in concentrations ν_1 and ν_2. Mathematically, this means applying Schoenberg-Muir (1989) calculus to the HTI media and produces the effective orthorhombic solid ORT2.

Perhaps, unexpectedly, the media ORT1 and ORT2 come out to be different. For example, their effective stiffness coefficients $\hat{c}_{66}$ are

$$\hat{c}_{66}^{\text{ORT1}} = \frac{\nu_1\mu_1 + \nu_2\mu_2}{1 + K_T(\nu_1\mu_1 + \nu_2\mu_2)} \tag{9}$$

and

$$\hat{c}_{66}^{\text{ORT2}} = \frac{\nu_1\mu_1}{1 + K_T\mu_1} + \frac{\nu_2\mu_2}{1 + K_T\mu_2}. \tag{10}$$

The equality $\hat{c}_{66}^{\text{ORT1}} = \hat{c}_{66}^{\text{ORT2}}$ is reached if and only if at least one of the following conditions is satisfied: $\nu_1 = 0$ and $\nu_2 = 1$, $\nu_1 = 1$ and $\nu_2 = 0$, $\mu_1 = \mu_2$, or $K_T = 0$.

Clearly, we would like to find out why any nontrivial combination of μ_i, ν_i $(i = 1, 2)$, and K_T leads to the two effective media that differ from each other. It would be also useful to understand which, if any, sequence of averaging procedures yields the correct effective solid. To analyze what went wrong, we need to recall the principles that make the Backus-type averages and the linear slip theory of Schoenberg (1980) work.

Explanation of the result

A horizontally layered medium with the z-axis directed across the layers allows Backus averaging because the components τ_{az} and ε_{bc} [where $a = (x, y, z)$ and $(b, c) = (x, y)$], of static stress and strain tensors are constant throughout the medium. The other stress and strain components are known to vary rapidly and have discontinuities at the medium interfaces. In contrast, the linear slip theory used to describe fractures assumes quite a different stress and strain behavior. It is based on the assumption that the stress tensor $\boldsymbol{\tau}(\mathbf{x})$ is constant in the vicinity of an infinitely thin fracture, whereas the strain components jump at the fracture faces. Specifically, ε_{xx} is discontinuous for fractures with normals pointing in the x-direction if $K_N \neq 0$.

Note that the requirements of Backus and linear slip theories are contradictory and cannot be satisfied simultaneously whenever the fractures intersect layer interfaces. Therefore, only the first round of averaging, which produces the VTI and HTI media above, can be applied. Since distributions of the stresses and strains in the corresponding layered and fractured media differ from those in the effective homogeneous VTI and HTI solids, the second averages that result in the effective media ORT1 and ORT2 cannot be legitimately performed. Thus, I conclude that, in general, both obtained solids ORT1 and ORT2 must have incorrect stiffness coefficients. [The effective stiffnesses $\hat{c}_{44}$ and $\hat{c}_{55}$ represent an exception. The coefficient $\hat{c}_{44}$ is not influenced by the presence of fractures (e.g., Schoenberg and Sayers, 1995); therefore, $\hat{c}_{44}^{\text{ORT1}} = \hat{c}_{44}^{\text{ORT2}}$. Explanation of the equality $\hat{c}_{55}^{\text{ORT1}} = \hat{c}_{55}^{\text{ORT2}}$ is given below.]

Finite-element modeling

To examine whether numerical modeling helps in explaining the above noticed inconsistencies, I performed computations for the model shown in Figure 1. For the boundary conditions

$$\boldsymbol{\tau} \cdot \mathbf{n}^{(1)}\Big|_{\substack{x=0\\x=1}} = [0, 0], \quad \boldsymbol{\tau} \cdot \mathbf{n}^{(3)}\Big|_{z=0} = [0, 1],$$
$$\boldsymbol{\tau} \cdot \mathbf{n}^{(3)}\Big|_{z=1} = [0, -1], \tag{11}$$

I computed the strain and stress fields with the finite-element method (Figure 2). In general, we observe that none of the strain or stress components is constant. This fact alone confirms an earlier conclusion that the linear slip theory and Backus average cannot be used to predict elastic properties of the effective solid.

Figure 2 reveals a number of interesting details. For instance, the field of $\varepsilon_{xx}(x, z)$ in Figure 2a shows that statement of the linear slip theory about discontinuities of strains at fracture faces is probably correct. Also, apart from the areas occupied by cracks and the ripples at model edges (which always exist due to heterogeneity), the strain $\varepsilon_{xx}(x, z)$ is constant in accordance with Backus theory.

Figures 2b and 2e illustrate the well-known fact of the presence of stress and strain concentrations around heterogeneities. Figures 2c and 2d display the fields of $\varepsilon_{zz}(x, z)$ and $\tau_{xx}(x, z)$, which vary rapidly in horizontally layered media without fractures. Indeed, the layer boundaries are clearly seen in those plots. In addition, the cracks are also clearly observed, especially in Figure 2c. The field of the stress component $\tau_{zz}(x, z)$ would be constant and equal to 1 in an uncracked medium (again, disregarding the edge effects) for boundary conditions (11). Instead, variation of $\tau_{zz}(x, z)$ in the vicinity of fractures by some 5–10% is observed.

Overall, Figure 2 shows a complex stress and strain distribution in the model. Since those complexities are not accounted for by the classical effective media theories, it is not surprising that they lead to contradictory and generally incorrect results.

I performed similar computations for the other two sets of boundary conditions specified by the tractions

$$\boldsymbol{\tau} \cdot \mathbf{n}^{(1)}\big|_{x=0} = [1, 0], \quad \boldsymbol{\tau} \cdot \mathbf{n}^{(1)}\big|_{x=1} = [-1, 0],$$
$$\boldsymbol{\tau} \cdot \mathbf{n}^{(3)}\big|_{\substack{z=0\\z=1}} = [0, 0]$$

and

$$\boldsymbol{\tau} \cdot \mathbf{n}^{(1)}\big|_{x=0} = [0, 1], \quad \boldsymbol{\tau} \cdot \mathbf{n}^{(1)}\big|_{x=1} = [0, -1],$$
$$\boldsymbol{\tau} \cdot \mathbf{n}^{(3)}\big|_{z=0} = [1,\ 0], \quad \boldsymbol{\tau} \cdot \mathbf{n}^{(3)}\big|_{z=1} = [-1,\ 0].$$

This allowed me to solve equations (6) and calculate the effective stiffness coefficients $\hat{c}_{11}^{\text{NUM}}$, $\hat{c}_{13}^{\text{NUM}}$, $\hat{c}_{33}^{\text{NUM}}$, and $\hat{c}_{55}^{\text{NUM}}$. Their values, along with those for the above discussed models ORT1 and ORT2, are given in Table 1. The coefficients $\hat{c}_{15}^{\text{NUM}}$ and $\hat{c}_{35}^{\text{NUM}}$ were also obtained; they turned out to be zero within numerical accuracy.

The differences between the stiffnesses computed by different methods are mild because the excess fracture compliances K_N and K_T used in this example are relatively small (see the caption to Figure 1). Other tests (not shown) indicate that those discrepancies increase for greater contrasts in Lamé constants of the layers and greater excess fractures compliances. Note that all methods give the same value for the effective stiffness coefficient $\hat{c}_{55}$ (Table 1). The mathematical reason for $\hat{c}_{55}^{\text{ORT1}} = \hat{c}_{55}^{\text{ORT2}}$ is the equality

$$\hat{c}_{55}^{\text{ORT1}} = \hat{c}_{55}^{\text{ORT2}} = \left\langle \frac{1}{\mu(z)} + K_T \right\rangle^{-1}. \tag{12}$$

It shows that both the linear slip and Backus theories operate with averages of exactly the same quantities—the compliances.

Table 1. Effective stiffness coefficients computed for the model in Figure 1.

Method of computation	Stiffness coefficients $\hat{c}_{11}$	$\hat{c}_{13}$	$\hat{c}_{33}$	$\hat{c}_{55}$
ORT1	6.92	3.14	7.84	1.89
ORT2	6.91	3.15	7.80	1.89
NUM	7.01	3.13	7.73	1.89

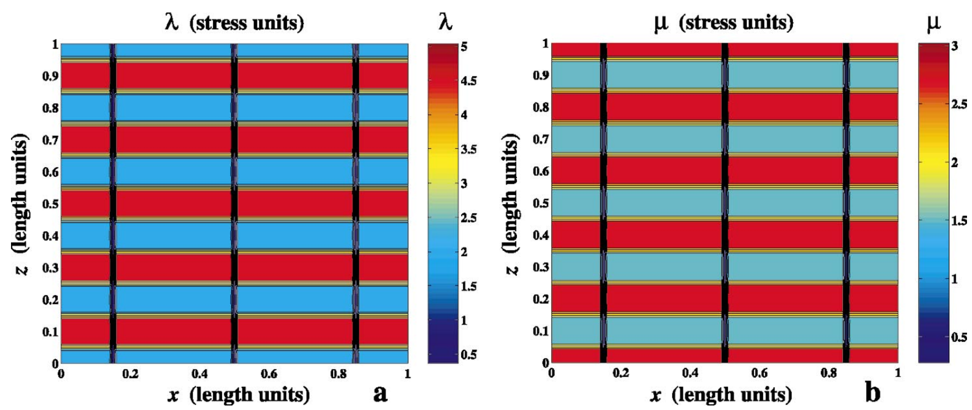

FIG. 1. Lamé constants (a) λ and (b) μ (in the units of stress) in layered medium containing throughgoing cracks. The fractures are modelled as thin soft vertical layers that correspond to the excess fracture compliances $K_N = 0.015$ and $K_T = 0.028$ (in the units reciprocal to stress). The color bars indicate values of λ and μ.

Interchanging the sequence of those averages leads just to rearrangement of terms in equation (12) and does not change $\hat{c}_{55}$. Our modeling gives the same numerical value for $\hat{c}_{55}^{\mathrm{NUM}}$, thus justifying the results of both effective media theories and validating the estimated computational error (which is presumably less than 0.25%).

LAYERED MEDIA WITH RANDOM INCLUSIONS

Next, I analyze a model that contains isotropic layers with random inclusions (Figure 3). Again, I compute the static stresses and strains (Figure 4). In the absence of inclusions, the effective medium would be given by Backus averaging that predicts constant fields of $\varepsilon_{xx}(x, z)$, $\varepsilon_{xz}(x, z)$, $\tau_{xz}(x, z)$, and $\tau_{zz}(x, z)$. Indeed, Figures 4a, 4b, 4e, and 4f indicate that these fields vary randomly around their average values and do not display any systematic patterns. (As before, I disregard the edge effects at $x < 0.2$ and $x > 0.8$.) Since one might expect this sort of behavior due to the influence of random inclusions, it is of interest to examine whether Backus theory can quantitatively predict the effective stiffness coefficients.

Table 2 compares the stiffnesses $\hat{c}_{ij}$ calculated using Backus average of the local Lamé constants λ and μ with those obtained by averaging the stresses and strains computed numerically (such as those shown in Figure 4) followed by applying definition (6). Backus average was performed for eleven vertical stripes of the model to find the mean values and standard deviations of the stiffness coefficients. We observe that while the coefficients $\hat{c}_{11}$ produced by the two averaging procedures coincide within one standard deviation, Backus average consistently underestimates the values of other stiffnesses.

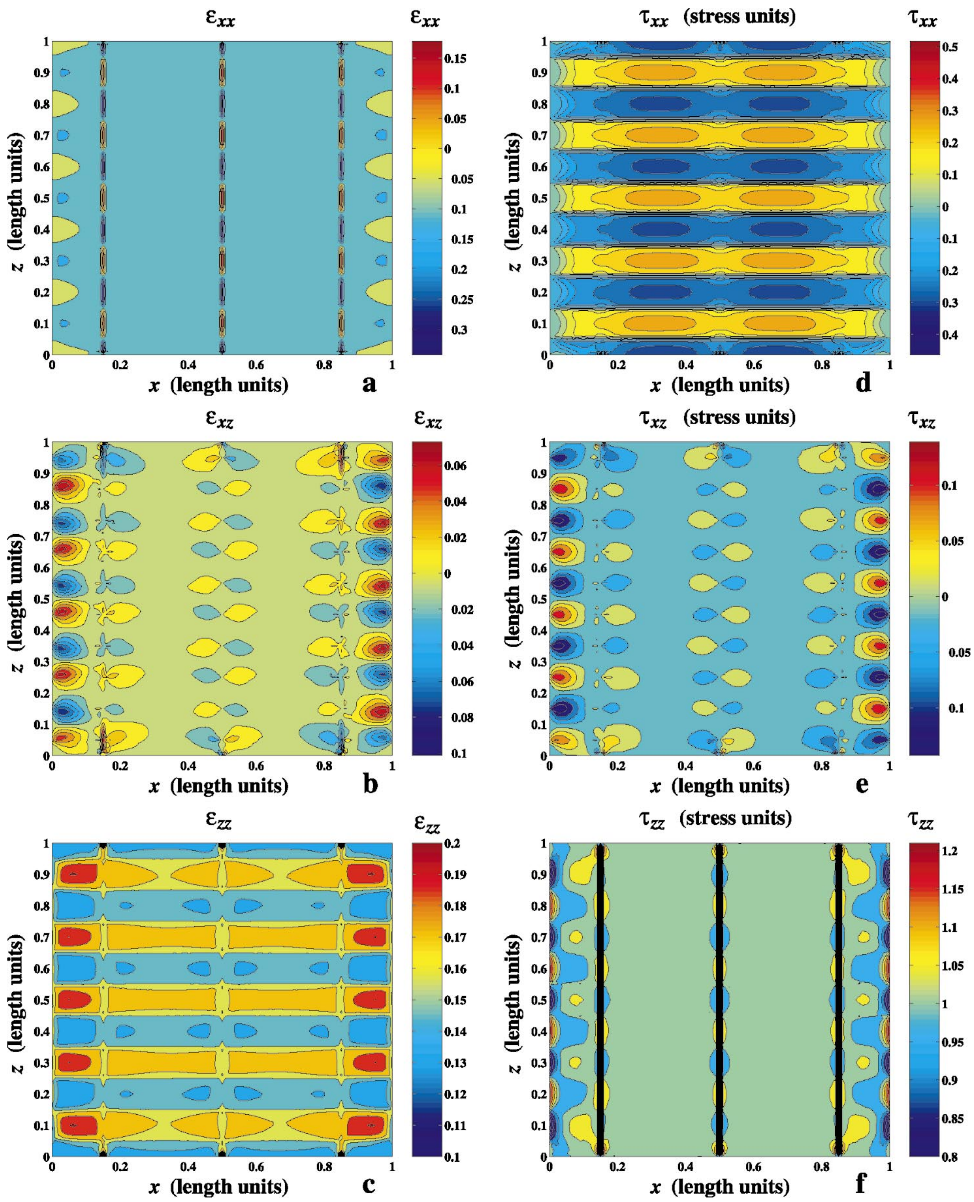

FIG. 2. (a)–(c) Strain and (d)–(f) stress fields in the model in Figure 1 computed for boundary conditions (11). Color bars represent the values of ε_{ab} and τ_{ab} $(a, b) = (x, z)$ on the adjacent panels.

Such a bias can be explained based on Backus (1962) expressions for the effective stiffness coefficients:

$$\hat{c}_{13} = \langle(\lambda + 2\mu)^{-1}\rangle^{-1}\left\langle\frac{\lambda}{\lambda + 2\mu}\right\rangle,$$
$$\hat{c}_{33} = \langle(\lambda + 2\mu)^{-1}\rangle^{-1}, \quad (13)$$
$$\hat{c}_{55} = \langle\mu^{-1}\rangle^{-1}.$$

They essentially represent the harmonic average

$$\check{f} = \langle[f(\mathbf{x})]^{-1}\rangle^{-1} \quad (14)$$

that has the following property: a tiny portion of sufficiently small values of the function $f(\mathbf{x}) \geq 0$ makes $\check{f}$ small. For instance, the average (14) of array $f(\mathbf{x}) = [0, \ 1, \ldots, \ 1]$ that has finite length L is identically 0, whereas $\langle f(\mathbf{x})\rangle$ approaches 1 as L grows.

Therefore, numerical modeling suggests that small-scale heterogeneities bias Backus-based estimates of the effective stiffnesses $\hat{c}_{13}$, $\hat{c}_{33}$, and $\hat{c}_{55}$ towards their smaller values. This conclusion has an important practical implication. Backus averaging applied for upscaling of sonic and shear logs (which presumably measure the true local velocities along a vertical borehole) will always result in too low effective vertical velocities because their possible lateral variations away from the borehole are not taken into account.

Table 2. Effective stiffness coefficients computed using Backus averaging and numerical modeling. The quantities $A \pm B$ (the mean ±standard deviation of the corresponding stiffness coefficient) are calculated applying Backus average to the local Lamé constants $\lambda(x, z)$ and $\mu(x, z)$ shown in Figure 3 in the overlapping vertical strips that have the width $\Delta x = 0.1$ (in length units) and are centered at $x_j = 0.25 + 0.5\, j\, \Delta\, x$, ($j = 0, \ldots, 10$).

Method of computation	Stiffness coefficients			
	$\hat{c}_{11}$	$\hat{c}_{13}$	$\hat{c}_{33}$	$\hat{c}_{55}$
Backus average	7.80 ± 0.02	3.46 ± 0.06	7.93 ± 0.02	2.05 ± 0.03
Numerical modeling	7.81	3.50	8.02	2.09

DISCUSSION

The main idea of this paper is straightforward: any numerical solver of the elastic equations of motion can be used as a tool for constructing effective media. This unconventional view allowed me to examine phenomena that have never been studied before. Most importantly, I showed the principal feasibility of quantitative evaluation of applicability of the existing effective media theories to a given microheterogeneous solid.

I presented two examples: a layered medium with through-going cracks and a layered solid containing random inclusions. In those examples, I intentionally violated certain assumptions of the effective media theories. As a result, the effective media computed numerically came out to be different from those predicted by the theories. I have found the following.

1) One cannot combine the linear slip theory and Backus average to put vertical fractures into a horizontally layered solid as long as the layer thicknesses are smaller or comparable to the length of fractures. It appears that those theories can be used in the only case when the fracture size is significantly smaller than the thickness of any individual layer.
2) Random inclusions presented in a horizontally layered solid produce the fields of static stresses and strains that resemble those predicted by Backus theory in the absence of inclusions. Nevertheless, Backus average applied to media containing inclusions inevitably leads to underestimation of the vertical P- and S-wave velocities.

Clearly, the above discussed examples have been selected to highlight the problems of standard effective media theories. Still, the differences in the effective stiffnesses $\hat{\mathbf{c}}$ computed using a variety of theories were found to be small. As analysis of Bakulin and Grechka (2003) indicates, this is always the case when heterogeneity is relatively mild. Conversely, the contrasts in elastic properties of a heterogeneous solid should be really large to create sizeable differences in the effective $\hat{\mathbf{c}}$ computed with different theories. Thus, in general, conventional effective media theories can be expected to work reasonably well in all weakly heterogeneous media.

It is also worthwhile to recall that my conclusions are based on finite-element computation of static deformations in 2D isotropic media. As one might guess, there are important properties of heterogeneous solids which can be modelled in

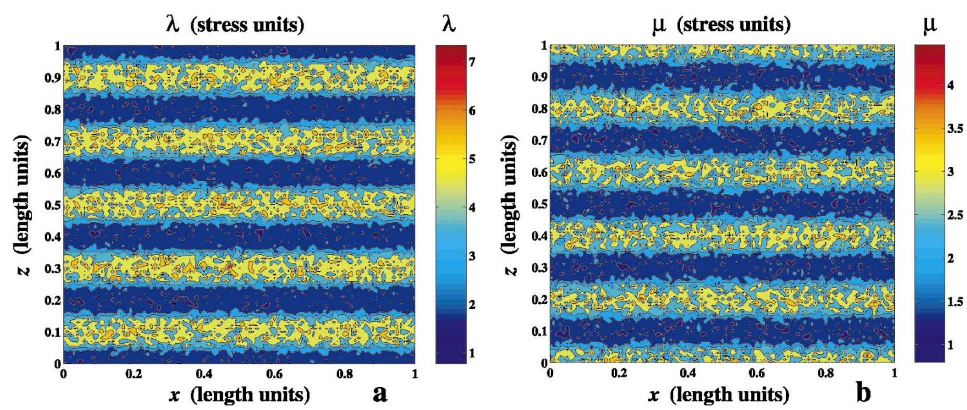

FIG. 3. Lamé constants λ and μ (in the units of stress) that represent a layered medium with random inclusions.

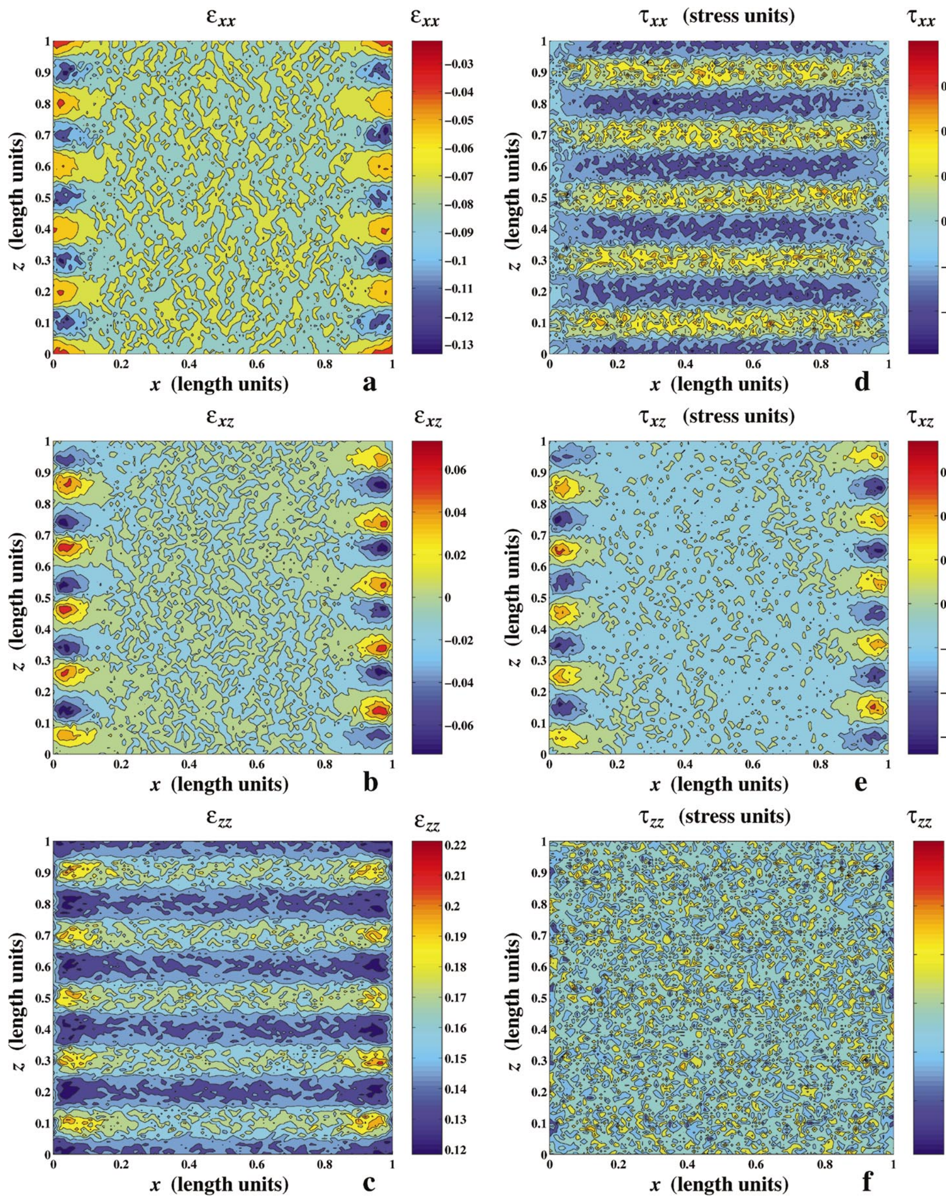

FIG. 4. Same as Figure 2 but for the model in Figure 3.

3D only. In particular, it would be possible to calculate the full effective stiffness tensor $\hat{\mathbf{c}}$ rather than its certain elements and make the original microheterogeneous solid fully anisotropic. One could also examine the influence of spatial rather than areal distribution, orientation, and shape of inclusions on $\hat{\mathbf{c}}$. Finally, since the equation of motion can be solved for any frequency, it becomes in principle possible to obtain frequency dependent effective stiffness tensor $\hat{\mathbf{c}}(\omega)$. Although some of these studies, [i.e., computing $\hat{c}_{ij}(\omega)$] can be also performed in 2D media, I leave them for future work.

ACKNOWLEDGMENTS

I thank Shell International E & P Inc. for permission to publish the paper. I am grateful to Andrey Bakulin (Shell) and Evgeni Chesnokov (University of Oklahoma) for valuable discussions and to anonymous reviewers for their critiques and suggestions.

REFERENCES

Backus, G. E., 1962, Long-wave elastic anisotropy produced by horizontal layering: J. Geophys. Res., **67**, 4427–4440.

Bakulin, A., and Grechka, V., 2003, Effective anisotropy of layered media: Geophysics, **68**, 1817–1821, this issue.

Bakulin, A., Grechka, V., and Tsvankin, I., 2000a, Estimation of fracture parameters from reflection seismic data. Part I: HTI model due to a single fracture set: Geophysics, **65**, 1788–1802.

——— 2000b, Estimation of fracture parameters from reflection seismic data. Part II: Fractured models with orthorhombic symmetry: Geophysics, **65**, 1803–1817.

——— 2000c, Estimation of fracture parameters from reflection seismic data. Part III: Fractured models with monoclinic symmetry: Geophysics, **65**, 1818–1830.

Christensen, R. M., 1979, Mechanics of composite materials: Wiley, New York.

Dellinger, J., Vasicek, D., and Sondergeld, C., 1998, Kelvin notation for stabilizing elastic-constant inversion: Revue de l'Institut Français du Pétrole, **53**, No. 5, 709–719.

Eshelby, J. D., 1957, The determination of the elastic field of an ellipsoidal inclusion and related problems: Proc. Roy. Soc. Lond., A, **241**, 376–396.

Helbig, K., 1984, Anisotropy and dispersion in periodically layered media: Geophysics, **49**, 364–373.
Hill, R., 1963, Elastic properties of reinforced solids; some theoretical principles: J. Mechanics and Physics of Solids, **11**, 357–372.
Hudson, J. A., 1980, Overall properties of a cracked solid: Math. Proc. Camb. Phil. Soc., **88**, 371–384.
——— 1991, Overall properties of heterogeneous material: Geophys. J. Internat., **107**, 505–511.
Rytov, S. M., 1956, Acoustical properties of a thinly laminated medium: Soviet Physical Acoustics, **2**, 67–80.
Schoenberg, M., 1980, Elastic wave behavior across linear slip interfaces: J. Acoust. Soc. Am., **68**, 1516–1521.
Schoenberg, M., and Douma, J., 1988, Elastic wave propagation in media with parallel fractures an aligned cracks: Geophys. Prosp., **36**, 571–590.
Schoenberg, M., and Helbig, K., 1997, Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth: Geophysics, **62**, 1954–1974.
Schoenberg, M., and Muir, F., 1989, A calculus for finely layered anisotropic media: Geophysics, **54**, 581–589.
Schoenberg, M., and Sayers, C. M., 1995, Seismic anisotropy of fractured rock: Geophysics, **60**, 204–211.