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On the impossibility of measuring the general relativistic part of the terrestrial acceleration of gravity with superconducting gravimeters

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SUMMARY

In this paper we very preliminarily investigate the possibility of measuring the post-Newtonian general relativistic gravitoelectric and gravitomagnetic components of the acceleration of gravity on the Earth, in continuous regime, with two absolute measurements at the equator and the South Pole with superconducting gravimeters. The magnitudes of such relativistic effects are 10^{-10} m s^{-2} and 10^{-11} m s^{-2}, respectively. Unfortunately, the present-day uncertainties in the Earth's geodetic parameters, which enter the classical Newtonian terms induce systematic errors that are one to two orders of magnitude larger than the relativistic ones. Moreover, a $\sim$1 ngal sensitivity can be reached by the currently available superconducting gravimeters, but only for relative measurements.

Key words: Earth-based experiments, post-Newtonian gravity, superconducting gravimeters.

1 INTRODUCTION

The classical test-bed of the Einstein's general theory of relativity (GTR), in its linearized weak-field and slow-motion approximation (Soffel 1989) valid in the Earth's neighbourhood and throughout the Solar System, has always been represented by the motion of test particles and electromagnetic waves in the gravitational field of massive astronomical bodies (Will 2001). This is so due to the extreme smallness of the relativistic effects with respect to the much larger Newtonian features of gravity.

However, in the more or less recent past various Earth-based laboratory experiments (Braginsky *et al.* 1984; Pippard 1988; Braginsky *et al.* 1977; Cerdonio *et al.* 1988; Tartaglia & Ruggiero 2002; Stedman *et al.* 2003; Iorio 2003) have been proposed, mainly to detect the elusive gravitomagnetic part of the gravitational field of the Earth induced by its proper angular momentum (see Section 2). Up to now, they have not been performed, mainly due to the extreme difficulty both to reach the necessary high sensitivity and to cope with the numerous and much larger classical noising effects.

In regard to the possibility of testing GTR with terrestrial geodetic techniques (Soffel 1989), gravimetry can allow to penetrate into the relativistic regime. Among the most sensitive apparatus there are the superconducting gravimeters (SG) (Goodkind 1999). Their most remarkable quality is their high immunity to the environmental temperature variations that generally determine variations of the elastic constant of the mechanical spring which, indeed, is replaced by a magnetic levitation of a superconducting sphere in the magnetic field of superconducting, persistent current coils. The goal is to utilize the high stability of such currents to create a perfectly stable spring. The magnetic levitation is designed to provide independent adjustment of the total levitating force so that it can support the full weight of the sphere and still yield a large displacement for a small change in gravity. Recent developments in such a field have pushed the sensitivity of SG down to $10^{-12}\ g = 10^{-11}$ m s^{-2} = 1 ngal (Francis & van Dam 2002) for relative measurements. For some SG performed tests of certain non-relativistic preferred-frame and preferred-locations effects, in the context of the parametrized post-Newtonian formalism, see Section 8.2 of Will (1993). However, it must also be noted that there are great difficulties to use SG in absolute measurements due to calibration issues.

2 THE POST-NEWTONIAN GENERAL RELATIVISTIC COMPONENTS OF THE ACCELERATION OF GRAVITY

The acceleration experienced by a test mass in the gravitational field of a central body, to the post-Newtonian $\mathcal{O}(c^{-2})$ level of GTR in its linearized weak-field and slow-motion approximation, is represented by a gravitoelectric component, due to the Schwarzschild part (Schwarzschild 1916) of the space–time metric, and a gravitomagnetic component, due to the Lense–Thirring part (Lense & Thirring 1918) of the space–time metric. We can write it as $a \equiv a_{\rm Newton} + a_{\rm Schwarzschild} + a_{\rm Lense-Thirring}$ with Soffel (1989)

$$\begin{cases} a_{\rm Newton} &= -\frac{GM}{r^3}r, \\ a_{\rm Schwarzschild} &= \frac{GM}{c^2r^3}\left\{\left[\frac{4GM}{r} - v^2\right]r + 4(r\cdot v)v\right\}, \\ a_{\rm Lense-Thirring} &= \frac{2G}{c^2r^5}\left[3(r\cdot J)r\times v + r^2 v\times J\right]. \end{cases} \quad (1)$$

G is the Newtonian gravitational constant, c is the speed of light in vacuum, M is the mass of the central body, J is its proper angular

Table 1. Relevant geodetic parameters.

Gravitational constant G (kg^{-1} m^3 s^{-2}) (Mohr & Taylor 1999)	$6.673 \times 10^{-11} \pm 1 \times 10^{-13}$
Speed of light c (m s^{-1}) (Mohr & Taylor 1999)	2.99792458×10^8
Earth's GM (m^3 s^{-2}) (Groten 1999)	$3.986004418 \times 10^{14} \pm 8 \times 10^5$
Earth's mean equatorial radius $R_{\rm eq}$ (m) (Groten 1999)	$6.3781366 \times 10^6 \pm 1 \times 10^{-1}$
Earth's angular momentum J (kg m^2 s^{-1}) (McCarthy 2004)	$5.85386532242 \times 10^{33}$
Earth's flattening f (Groten 1999)	$3.352 \times 10^{-3} \pm 1.1 \times 10^{-10}$
Earth's angular speed ω (rad s^{-1}) (Groten 1999)	$7.292115 \times 10^{-5} \pm 1 \times 10^{-12}$

momentum,[1] $\boldsymbol{r}$ and $\boldsymbol{v}$ are the position and velocity vectors, respectively, of the test mass. It is assumed that the origin of the adopted inertial reference frame is located at the centre of mass of the body of mass M.

3 ABSOLUTE MEASUREMENTS AT THE POLE AND THE EQUATOR

We will now explore the possibility of measuring the gravitoelectric and gravitomagnetic components of the terrestrial gravitational field by measuring the acceleration of gravity at the equator and at the South Pole and making a crosscheck between these two absolute measurements. To this aim, let us, now, consider a test body fixed on the Earth's surface, so that $\boldsymbol{v} = \boldsymbol{\omega} \times R\hat{\boldsymbol{r}}$ and $v = \omega R \cos\lambda$, where $\boldsymbol{\omega} = \omega \boldsymbol{k}$ is the Earth's angular velocity vector, $\hat{\boldsymbol{r}}$ is a unit vector from the Earth's centre to the location of the test mass, λ is the geocentric latitude and R is the Earth's radius at the location of the test body. From eq. (1) it is easy to see that at the equator ($\boldsymbol{r} \cdot \boldsymbol{J} = 0$) the Lense–Thirring acceleration is entirely radial and directed outwards, as the centrifugal acceleration, while at the South Pole it is absent ($\boldsymbol{v} = 0$). The gravitoelectric Schwarzschild component is always directed radially because at the equator $\boldsymbol{r} \cdot \boldsymbol{v} = 0$ and at the poles $\boldsymbol{v} = 0$. Then, the acceleration of gravity can be written as

$$\begin{cases} g_{\rm eq} = \frac{GM}{R_{\rm eq}^2} - \omega^2 R_{\rm eq} + \frac{GM}{c^2 R_{\rm eq}^3}\left(4GM - \omega^2 R_{\rm eq}^3\right) - \frac{2GJ\omega}{c^2 R_{\rm eq}^2}, \\ g_{\rm pol} = \frac{GM}{R_{\rm pol}^2} + \frac{4(GM)^2}{c^2 R_{\rm pol}^3}, \end{cases} \tag{2}$$

where the difference between the Earth's radius at the poles and at the equator is related to the Earth's flattening f by

$$\frac{R_{\rm eq} - R_{\rm pol}}{R_{\rm eq}} \equiv f. \tag{3}$$

Then, we can write for $\Delta g \equiv g_{\rm pol} - g_{\rm eq}$

$$\Delta g \simeq \frac{2GM}{R_{\rm eq}^2}\frac{f}{(1-f)^2} + \frac{12(GM)^2}{c^2 R_{\rm eq}^3}\frac{f}{(1-f)^3} + \omega^2\left(R_{\rm eq} + \frac{GM}{c^2}\right) + \frac{2GJ\omega}{c^2 R_{\rm eq}^2}. \tag{4}$$

Now we will evaluate the magnitude of the various terms entering eq. (4) in order to compare them with the present-day available sensitivity of SG. Then, we will also evaluate the systematic errors induced by the classical terms of eq. (4) due to the uncertainties in the various geodetic parameters of the Earth. Of course, there are also various classical time-dependent competing surface gravity effects[2] spanning a wide range of periodicities from 1 s to more than 1 year and magnitudes up to 1–10 μgal $= 10^{-5} - 10^{-4}$ m s^{-2} which would act as systematic bias.[3] They should be accounted for in a detailed error budget, which is, however, beyond the scope of the present paper.

3.1 The Lense–Thirring component

By using the values of Table 1 it turns out that the Lense–Thirring component $\Delta g_{\rm LT} \equiv 2GJ\omega/c^2 R_{\rm eq}^2$ of (4) amounts to 1.5×10^{-11} m s^{-2}, that is, $\sim$1 ngal.

3.1.1 The systematic errors

In order to obtain $\Delta g_{\rm LT}$ we should subtract the first three terms of eq. (4) from the measured Δg. This can only be done if the uncertainty of the terms to be subtracted is smaller that the predicted value of the Lense–Thirring component.

The residual Schwarzschild component would not pose problems. Indeed, its nominal value is

$$\Delta g_{\rm Schwarzschild} = \frac{12(GM)^2}{c^2 R_{\rm eq}^3}\frac{f}{(1-f)^3} + \frac{GM}{c^2}\omega^2 = 3.00 \times 10^{-10}\ {\rm m\ s^{-2}}. \tag{5}$$

According to the uncertainties released in Table 1 it turns out that $\delta(\Delta g_{\rm Schwarzschild}) \ll \Delta g_{\rm LT}$. The problems come from the classical terms. Indeed,

$$\Delta g_{\rm Newton} = \frac{2GM}{R_{\rm eq}^2}\frac{f}{(1-f)^2} = 6.6130308646 \times 10^{-2}\ {\rm m\ s^{-2}}, \tag{6}$$

which is nine orders of magnitude larger than the gravitomagnetic term. The errors induced by the uncertainty in f and the Earth's mean equatorial radius $R_{\rm eq}$ are the largest ones and amount to

$$\delta(\Delta g_{\rm Newton})|_f = \frac{2GM}{R_{\rm eq}^2}\frac{(1+f)}{(1-f)^3}\delta f = 2.184 \times 10^{-9}\ {\rm m\ s^{-2}} \tag{7}$$

and

$$\delta(\Delta g_{\rm Newton})|_{R_{\rm eq}} = \frac{4GMf}{R_{\rm eq}^3(1-f)^2}\delta R_{\rm eq} = 2.073 \times 10^{-9}\ {\rm m\ s^{-2}}, \tag{8}$$

respectively. The centrifugal component amounts to

$$\Delta g_{\rm centrifugal} = \omega^2 R_{\rm eq} = 3.3906727993 \times 10^{-2}\ {\rm m\ s^{-2}}, \tag{9}$$

with an error of $\delta(\Delta g_{\rm centrifugal}) = 1.48 \times 10^{-9}$ m s^{-2}. The total error in the classical part of Δg is, thus

$$\delta(\Delta g)_{\rm total}^{\rm class} \leq 5.737 \times 10^{-9}\ {\rm m\ s^{-2}}; \tag{10}$$

it is two orders of magnitude larger than $\Delta g_{\rm LT}$. This rules out the possibility of measuring the gravitomagnetic Lense–Thirring component of the acceleration of gravity with the present-day SG.

[1] For a homogeneous spherical body of mass M, radius R spinning at a rate ω $J = (2/5)MR^2\omega$; for the Earth the factor $2/5 = 0.4$ is replaced by 0.330841 (McCarthy 2004).

[2] They are, foe example, the ocean noise, seismic and normal modes, slow and silent earthquakes, secular deformations.

[3] It must be noted that the investigated measurement would be in continuous regime.

3.2 The Schwarzschild component

From (4) it can be noted that an absolute measurement at the South Pole could allow to detect the gravitoelectric Schwarzschild component of the acceleration of gravity. Indeed, it turns out that

$$g_{\rm pol}^{\rm Schwarzschild} = \frac{4(GM)^2}{c^2 R_{\rm eq}^3 (1-f)^3} = 2.752 \times 10^{-8}\ {\rm m\ s^{-2}}. \quad (11)$$

3.2.1 *The systematic errors*

On the other hand, the errors in the classical part are $\delta g_{\rm pol}^{\rm class}|_{GM} = 1.979 \times 10^{-8} {\rm m\,s^{-2}}$, $\delta g_{\rm pol}^{\rm class}|_{R_{\rm eq}} = 3.0724 \times 10^{-7} {\rm m\,s^{-2}}$, $\delta g_{\rm pol}^{\rm class}|_{f} = 1.97 \times 10^{-9} {\rm m\,s^{-2}}$. The total systematic error would then be

$$\delta g_{\rm pol}^{\rm class}|_{\rm total} = 3.29 \times 10^{-7}\ {\rm m\ s^{-2}}, \quad (12)$$

that is, one order of magnitude larger than $g_{\rm pol}^{\rm Schwarzschild}$. Note that eqs (5) and (10) rule also out the possibility of a measurement of the gravitoelectric component of the acceleration of gravity between the pole and the equator by one order of magnitude.

4 DISCUSSION AND CONCLUSIONS

In this paper we have preliminarily investigated the feasibility of an experiment aimed at the measurement of the general relativistic gravitoelectric and gravitomagnetic components of the terrestrial acceleration of gravity at the equator and at the pole. It turns out that they are quite small; the difference between the gravitoelectric accelerations in the two places amounts to 3×10^{-10} m s^{-2} and the difference of the gravitomagnetic accelerations is 1.5×10^{-11} m s^{-2}, that is, 10 ngal and 1 ngal, respectively. Such absolute measurements could be done, in principle, with SG. However, it must be noted that the greatest difficulty in the use of the SG as an absolute instrument is connected with its calibration. Another non-trivial difficulty is that the effects to be measured are in continuous. Moreover, the present-day sensitivity of SG is just of the order of ngal, but only for relative measurements. In regard to systematic bias, the current uncertainties in the Earth's geodetic parameters, which enter the classical Newtonian terms to be subtracted, induce errors one to two orders of magnitude larger than the relativistic ones.

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