Viscosity of the Earth's fluid core and torsional oscillations

Jon Mound[1,2] and Bruce Buffett[3]

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[1] Viscous effects on the temporal behavior and spatial structure of torsional oscillation normal modes of the Earth's fluid core are examined. If the viscosity of the fluid core is equal to that inferred from theoretical and experimental studies of liquid metals, then viscous effects on torsional oscillations are unlikely to be important; however, if the viscosity is as high as allowed from observational constraints, then Ekman layers at the fluid core boundaries may play an important role in damping torsional oscillations. The observationally inferred decay time of decadal torsional oscillations leads to an upper bound on the viscosity of the fluid core at the core-mantle boundary of the order of 10^{-2} m^2/s. A sufficiently large viscosity would result in torsional oscillation normal modes that undergo pure decay, as opposed to damped oscillatory motion. The viscosity value at which the temporal behavior switches between these regimes depends on the period of the normal mode. Numerical geodynamo models must reach an Ekman number of the order of 10^{-7} for free-slip boundaries or 10^{-10} for no-slip boundaries to accurately model the temporal behavior of decadal period torsional oscillations.

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1. Introduction

[2] The viscosity of the Earth's fluid core remains one of the most poorly constrained geophysical parameters, with published estimates ranging over 14 orders of magnitude (for compilations of viscosity estimates based on a wide variety of techniques see *Lumb and Aldridge* [1991] and *Secco* [1995]). The estimates can be roughly divided into two categories: those based on theoretical or laboratory studies of liquid metals; and those based on geophysical observations. The liquid metal studies generally indicate values for the kinematic viscosity, ν, of the Earth's fluid core that are several orders of magnitude lower than those obtained from observational constraints. For example, recent analyses suggest a viscosity of $o\{10^{-2}\}$ m^2/s may be required to explain the measured dissipation of the Earth's nutations [*Mathews and Guo*, 2005; *Palmer and Smylie*, 2005; *Deleplace and Cardin*, 2006], a value at the low end of observational constraints; whereas a viscosity of $o\{10^{-6}\}$ m^2/s [e.g., *Gans*, 1972; *de Wijs et al.*, 1998; *Rutter et al.*, 2002] is typical of the liquid metal results.

[3] The observational constraints are generally upper bounds and thus the disagreement between these two categories of viscosity estimates is not as great as it might first appear; the liquid metal value referenced above is typically taken as representative of the Earth's fluid core [e.g., *Bloxham*, 1998; *Dormy et al.*, 2000; *Hollerbach*, 2003]. Another possibility is that rather than the molecular viscosity value obtained from liquid metal studies the observationally inferred viscosity values represent an effective, "turbulent" viscosity value. Increased effective viscosity due to convective turbulence has been measured in laboratory analogue models [*Brito et al.*, 2004] and theoretical arguments suggest that the turbulent viscosity value for the Earth's fluid core may be as large as $o\{10^{0}\}$ m^2/s [*Braginsky and Meytlis*, 1990]. However, even this turbulent viscosity value is too small to be attainable in current numerical geodynamo models [see, e.g., *Dormy et al.*, 2000].

[4] In this paper we investigate the influence of viscous effects on the behavior of torsional oscillations. Torsional oscillations involve cylinders of fluid aligned with the planetary rotation axis (Taylor cylinders) undergoing rigid body rotations in response to torques that occur because of departures of the core from the so-called Taylor state [*Taylor*, 1963]. The magnetic field threading through adjacent Taylor cylinders couples their motion and results in angular momentum transfer throughout the fluid core via the propagation of cylindrical shear waves travelling at the shear Alfvén wave speed; reflection and interference produce standing waves that are termed torsional oscillations [e.g., *Braginsky*, 1970]. (The term torsional oscillation is

[1]Department of Earth and Planetary Sciences, Harvard University, Cambridge, Massachusetts, USA.
[2]Now at School of Earth and Environment, University of Leeds, Leeds, UK.
[3]Department of Geophysical Sciences, University of Chicago, Chicago, Illinois, USA.

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also used in the literature to denote the general phenomenon of magnetically coupled, cylindrical, shear waves in the fluid core, and in the remainder of this work references to the normal modes in specific will be indicated.)

[5] Coupling between the fluid core and the surrounding solid bodies arises because of differential motion across the core-mantle boundary (CMB) and inner core boundary (ICB); this coupling implies that the torsional oscillation normal modes include associated rotations of the mantle and solid inner core. The motion of the mantle associated with torsional oscillations is observed as variations in the length of day (LOD) [e.g., *Jault et al.*, 1988; *Jackson et al.*, 1993]. The torque required to produce the observed change in the mantle's angular momentum is on the order of 10^{18} N m and the ability of electromagnetic, topographic and gravitational coupling mechanisms to supply this torque have been investigated (see, e.g., *Kuang and Chao* [2003] and *Ponsar et al.* [2003] for a review). The theory describing the influence of viscosity on torsional oscillations has been developed by *Braginsky* [1984]; however, viscous effects have generally been ignored in torsional oscillation studies because of the presumed low viscosity of the fluid core [e.g., *Braginsky*, 1970, 1984; *Buffett*, 1998; *Mound and Buffett*, 2003]. We seek to determine for what range of viscosity values the neglect of viscosity in torsional oscillation studies is a justified assumption.

[6] Numerical geodynamo models have been used to study torques within the core-mantle system [e.g., *Kuang and Chao* 2003] and on Taylor cylinders [*Dumberry and Bloxham*, 2003]. However, because of computational constraints, such models cannot be run at Earth-like fluid viscosities. The Ekman number, $E = \nu/2\Omega r_f^2$, provides a nondimensional measure of the importance of viscosity relative to the Coriolis force (Ω is the planetary rotation rate and r_f the radius of the fluid core). On the basis of the viscosity suggested by liquid metal studies the Ekman number for the Earth's fluid core is $o\{10^{-16}\}$; the numerical studies mentioned above [*Kuang and Chao*, 2003; *Dumberry and Bloxham*, 2003] were run at Ekman numbers of 1.25×10^{-6} and 2×10^{-5}, respectively.

[7] Viscous effects within the bulk of the fluid, at the CMB, and at the ICB will damp torsional oscillations. The dissipation arising from a high fluid viscosity would be expected to shorten the decay times and lengthen the periods of torsional oscillation normal modes. A sufficiently large viscosity would prevent free oscillations and allow only exponentially decaying modes. Numerical geodynamo models employ relatively high-viscosity values; therefore it may be necessary to carefully consider the influence of viscosity when numerical results are interpreted in terms of torsional oscillations of the Earth. Viscous dissipation within the bulk of the fluid scales as $o\{E\}$, whereas the scaling of dissipation in boundary layers depends on the assumed boundary conditions. Geodynamo models have employed free-slip, "partially slippery" and no-slip conditions at the boundaries of the fluid core. With free-slip boundary conditions there is no viscous coupling to the surrounding solids, the only viscous effects are those within the bulk of the fluid; this helps to maintain a reasonable torque balance on Taylor cylinders [*Kuang and Bloxham*, 1997; *Kuang*, 1999]. Partially slippery boundary conditions result in torques between the fluid and solid regions that are $o\{E\}$, whereas no-slip boundary conditions result in torques of $o\{\sqrt{E}\}$ [*Kuang and Chao*, 2003]. In this work we consider the end-member cases of free-slip and no-slip conditions at the boundaries of the fluid core.

[8] Estimates of viscosity from liquid metal studies [e.g., *Gans*, 1972] suggest a very small Ekman number for the fluid core. Therefore viscosity is unlikely to prevent free torsional oscillation normal modes in the Earth. Constraints on the flow at the surface of the core can be obtained from observations of geomagnetic secular variation [e.g., *Jault et al.*, 1988; *Bloxham and Jackson*, 1991; *Pais and Hulot*, 2000]. Models of core flow have been found to contain wave-like motions that are interpreted to be the signature of torsional oscillations. *Hide et al.* [2000] produced models of core angular momentum containing waves propagating from low to high latitudes, consistent with an underdamped torsional oscillation system being excited by a low-latitude forcing. However, the model of *Hide et al.* [2000] is also consistent with an overdamped torsional oscillation system being excited by convective flow with a preferred variability; in which case the observed, dominant ~65 year period in their model would be unrelated to torsional oscillation normal modes. *Zatman and Bloxham* [1997, 1998] resolved two waves in core flow models, interpreted as decadal torsional oscillation normal modes with decay times equal to, or somewhat exceeding, their periods. If such models do represent the decay of underdamped torsional oscillation normal modes, then the inferred decay times provide a constraint on the dissipation of torsional oscillations that is independent of the assumed dissipation mechanism. Although electromagnetic coupling can provide the required level of dissipation, by assuming that viscous effects are the sole means of damping torsional oscillations we obtain an upper bound on the viscosity of the fluid core.

2. Theory

[9] We begin with a review of the equations governing torsional oscillations, including the manner in which viscous effects can be included. Torsional oscillations involve rigid body rotations of fluid cylinders in the outer core and associated motions of the mantle and inner core. The fluid cylinders are coupled to each other by the magnetic field that threads through them; coupling to the surrounding solid regions may occur via magnetic, topographic, viscous or gravitational mechanisms. Following from the conservation of angular momentum, the angular velocity perturbation of the fluid, $u_f(s,t)$, can be described by [e.g., *Braginsky*, 1970, 1984]

$$m(s)\frac{\partial^2 u_f}{\partial t^2} = \frac{\partial}{\partial s}\left(\tau(s)\frac{\partial u_f}{\partial s}\right) - \frac{\partial F}{\partial t}, \tag{1}$$

where

$$m(s) = 4\pi\rho s^3\left(z_f - z_i\right) \tag{2}$$

is the moment density of the fluid cylinders and

$$\tau(s) = 4\pi\frac{\{B_s^2\}}{\mu}s^3\left(z_f - z_i\right) \tag{3}$$

Table 1. Typical Physical Parameters Used in Calculations

Parameter	Symbol	Value
Radius of ICB	r_i	1.22×10^6 m
Radius of CMB	r_f	3.48×10^6 m
Fluid core density	ρ	1.2×10^4 kg m^{-3}
Moment of inner core	C_i	5.87×10^{34} kg m^2
Moment of mantle	C_m	7.12×10^{37} kg m^2
Average radial magnetic field in fluid	B_s	0.2 mT
Peak magnetic field across ICB	B_i	3.5 mT
Peak magnetic field across CMB	B_m	0.5 mT
Core conductivity	σ_f	5×10^5 m^{-1}
Mantle conductance	$\sigma_m\Delta$	10^8 S
Gravitational coupling constant	$\tilde{\Gamma}_g$	3×10^{20} N m

is the magnetic tension. The resultant torsional oscillation normal modes depend on physical properties of the core such as the fluid density, ρ, and the (cylindrically) radial component of the magnetic field, B_s, and on the geometry of the core, $z_f = \sqrt{r_f^2 - s^2}$ and $z_i = \sqrt{r_i^2 - s^2}$ being the positions of the CMB and ICB respectively, in cylindrical coordinates. However, *Mound and Buffett* [2003] have shown that the influence of the core's geometry on the behavior of relatively short-period modes is small. Coupling between the fluid core and the surrounding solid regions will also influence the temporal and spatial form of the normal modes and is included in equation (1) through a "friction" on the cylinder ends, $F(s,t)$, which may represent any of the possible fluid-solid coupling mechanisms.

[10] We will consider both magnetic and viscous couplings between the fluid core and the surrounding solid regions. If the magnetic fields at the CMB and ICB are assumed to be axial dipoles with peak strengths B_m and B_i, respectively, then the magnetic couplings at the CMB and ICB are described by [*Buffett*, 1998]

$$F_m^b(s,t) = 4\pi\sigma_m\Delta B_m^2 s^3\left(\frac{z_f}{r_f}\right)\left(u_f - u_m\right), \tag{4}$$

$$F_i^b(s,t) = \pi(1+i)\sigma_f\delta_f B_i^2 s^3\left(\frac{z_i}{r_i}\right)\left(u_f - u_i\right). \tag{5}$$

Where, $\sigma_m\Delta$ is the total mantle conductance (a mantle conductivity that decreases exponentially from the CMB value of σ_m is assumed), σ_f is the fluid core conductivity, δ_f is the skin depth of the fluid core, and $u_m(t)$ and $u_i(t)$ are the angular velocity perturbations of the mantle and inner core, respectively. The conductance of the lowermost mantle is not directly observable; however, in those cases where CMB electromagnetic coupling is included we adopt a value that is compatible with constrains from studies of the Earth's nutations [e.g., *Mathews et al.*, 2002].

[11] The assumption of an axial, dipolar magnetic field at the CMB and ICB is mathematically convenient and is sufficient for the presence purpose of obtaining an order of magnitude estimate of the electromagnetic torques. However, the magnetic field of the Earth is more complicated and our approximation does neglect certain dynamics of the core-mantle system. The interaction between the assumed core flow and a more complicated magnetic field will produce both axial and equatorial torques on the inner core and mantle, resulting in polar motion coupled to LOD variations [*Dumberry and Bloxham*, 2002; *Mound*, 2005; *Nakada*, 2006]. However, this coupled motion has not been established through observation and in this study we focus solely on comparisons with the observed axial motions of the core and mantle (torsional oscillations and LOD variations).

[12] A velocity difference between the bulk of the fluid and the mantle will result in the formation of an Ekman boundary layer at the CMB which will produce a viscous coupling [*Braginsky*, 1984]

$$F_m^v(s,t) = 4\pi\rho\left(\nu\Omega\frac{r_f}{z_f}\right)^{1/2} s^3\left(u_f - u_m\right). \tag{6}$$

A similar expression can be generated for the viscous coupling with the solid core by the exchange of the equivalent variables in equation (6) with r_i, z_i and u_i.

[13] The relative strengths of electromagnetic and viscous couplings at the CMB and ICB can be gauged by considering the ratios $|F_m^b/F_m^v|$ and $|F_i^b/F_i^v|$. Substitution of typical values (see Table 1) indicates that magnetic and viscous torques at the CMB will be equal when $\nu \approx 0.01$ m^2/s, at the ICB the torques are equal when $\nu \approx 80{,}000$ m^2/s. The strong electromagnetic coupling at the ICB causes the fluid within the tangent cylinder to become "locked" to the inner core (see, e.g., *Braginsky* [1984] and *Buffett and Mound* [2005] and the normal mode structures presented below). Furthermore, dissipation due to relative motion across the ICB is dominantly electromagnetic for the range of viscosities that we will consider.

[14] The effect of viscosity within the bulk of the fluid core can also be included in equation (1). When viscosity within the fluid is included the torsional oscillation equation becomes [*Braginsky*, 1984]

$$m(s)\frac{\partial^2 u_f}{\partial t^2} = \frac{\partial}{\partial s}\left(\left(\tau(s) + V(s)\frac{\partial}{\partial t}\right)\frac{\partial u_f}{\partial s}\right) - \frac{\partial F}{\partial t}, \tag{7}$$

where

$$V(s) = 4\pi\rho\nu s^3\left(z_f - z_i\right). \tag{8}$$

Note that internal viscous effects scale directly with viscosity, and hence E (see equation (8)), whereas viscous effects at the no-slip boundaries of the fluid core scale as the square root of viscosity, i.e., as $\sqrt{E}$ (see equation (6)). Therefore the effect of viscosity on the behavior of the normal modes will depend on the relative importance of internal and boundary layer viscous processes.

[15] Mantle-inner core gravitational (MICG) coupling is also considered in this work. Because of aspherical density distributions in the mantle and inner core there will be a orientation for which the two bodies have a minimum gravitational potential energy [*Buffett*, 1996a]. Rotation away from this equilibrium position results in a restoring torque [*Buffett*, 1996b]

$$\Gamma = \tilde{\Gamma}_g\Delta\phi, \tag{9}$$

where $\Delta\phi$ is the relative, rotational, misalignment of the mantle and inner core. The value of $\tilde{\Gamma}_g$ can be estimated from seismically constrained models of mantle convection [*Buffett*, 1996a] or from an observed oscillation in LOD that has been interpreted to be the signature of the MICG normal mode [*Mound and Buffett*, 2006]. On the basis of these considerations we adopt a value of $\tilde{\Gamma}_g = 3 \times 10^{20}$ N m.

[16] The normal modes of this model of the core-mantle system are found numerically using the approach outlined in *Buffett and Mound* [2005]. Unlike numerical geodynamo models, the torsional oscillation model can be solved for the parameter regime expected for the Earth's core. However, the results of this study may not be directly applicable to the Earth's core because of one key simplification that has not yet been mentioned, the neglect of magnetic diffusion in the interior of the fluid core. Magnetic diffusion is included in the boundary layers at the CMB and ICB where large gradients in the magnetic field are produced by shearing of the field as fluid moves relative to the adjacent, conductive solid. However, torsional oscillations will also be damped by diffusion of the magnetic perturbations created by the oscillations within the bulk of the fluid core; this effect has been neglected in the model.

[17] Neglect of magnetic diffusion within the fluid interior is justified provided that the periods of the torsional oscillation normal modes are short in comparison to the associated magnetic diffusion timescale, $\tau_\eta = \lambda^2/\eta$. Although the normal modes do not have strictly sinusoidal forms (see the Results section below) a characteristic wavelength for the n^{th} mode can be defined as $\lambda_n = \frac{2}{n}(r_f - r_i)$. The period of the mode is thus approximately $T = \lambda_n/c$; where $c = \sqrt{B_s^2/\rho\mu}$ is the shear Alfvén wave speed. The period of the normal modes thus depends on the strength of the magnetic field within the fluid core. The Lundquist number for the n^{th} mode (i.e., the ratio of τ_η to the period) is

$$\left(\frac{\tau_\eta}{T}\right)_n \approx \frac{\frac{2}{n}\left(r_f - r_i\right)c}{\eta}. \tag{10}$$

Taking a value of $\eta \simeq 2$ m^2/s for the magnetic diffusivity of the fluid core [e.g., *Bloxham*, 1998] and a value of $B_s = 0.2$ mT, normal modes with periods of two years or more (approximately the first 45 modes) will have Lundquist numbers of order 100–1000. Therefore the neglect of magnetic diffusion within the bulk of the fluid core is, in general, justified in studies of interannual and decadal torsional oscillations. In this study we restrict our investigation to normal modes with periods of at least 2 years so that the neglect of internal magnetic diffusion is justified; magnetic diffusion in the boundary layers is incorporated through equations (4) and (5).

[18] The importance of viscosity relative to magnetic diffusion is reflected by the magnetic Prandtl number, $P_m = \nu/\eta$. The Earth is expected to have P_m of $o\{10^{-6}\}$ for molecular values of viscosity, but of $o\{10^0\}$ for turbulent viscosity values [e.g., *Braginsky and Roberts*, 1995; *Bloxham*, 1998]. Geodynamo models generally consider P_m values of $o\{10^0\}$ or greater [*Dormy et al.*, 2000]. In this work we always neglect the effects of magnetic diffusion within the bulk of the fluid core; however, the effects of magnetic diffusion in the boundary layers near the CMB and ICB are considered. When comparing this study to observations of the Earth or to numerical results it is important to consider the implications of the different parameter regimes.

[19] Geodynamo models consider relatively large viscosities. Regardless of future improvements in computational abilities it will be necessary to include viscous effects in any torsional oscillation model that is to be compared to geodynamo results. For the model developed here it may be advantageous to make comparisons with geodynamo models in which both E and P_m are large, as magnetic diffusion within the bulk of the fluid would be expected to have only a small effect on torsional oscillations in such cases. Comparisons with specific geodynamo results would require the inclusion of forcing terms to the model presented here, a problem which has been investigated in previous work [*Buffett and Mound*, 2005]. It would also be necessary to account for the lengthening of the timescale of torsional oscillations that occurs because of the large value of η used in numerical geodynamo models [*Dumberry and Bloxham*, 2003]. Despite these caveats, if geodynamo models do indeed contain torsional oscillations, then such motions should be compatible with a (properly modified) model similar to the one discussed in this work.

[20] For torsional oscillation models with small viscosity values (i.e., small P_m), which are most likely to be relevant to the Earth's core, it must be remembered that magnetic diffusion in the bulk of the fluid would increase the dissipation of the normal modes. The effect of magnetic diffusion in the bulk of the fluid is roughly equal to the effect of the magnetic boundary layers for the shortest-period normal modes considered in this study. However, for longer-period normal modes dissipation due to magnetic diffusion is dominated by boundary layer effects. For example, we find that for normal modes with periods near 4 years dissipation in the boundary layers is approximately ten times greater than that expected from internal magnetic dissipation. Therefore the inclusion of magnetic diffusion in the bulk of the fluid would have little effect on the main results presented below.

3. Results

[21] In this section we present results obtained from three variations of the torsional oscillation model described above. Our base model does not include any viscous effects, we refer to this as the viscosity free case. The next model, termed the internal viscosity case, includes the effect of viscosity within the bulk of the fluid core but not at the ICB and CMB. Finally, the full-viscosity case includes both internal viscous effects and the effect of the Ekman layers at the CMB and ICB. For all three models we consider cases both with and without CMB electromagnetic and MICG couplings. Neglecting these couplings allows us to isolate the effect of viscosity on the normal modes, whereas their inclusion allows us to investigate the importance of viscous dissipation relative to other mechanisms that are likely to be important in the Earth. For simplicity, we initially consider cases in which B_s is constant (models with variable B_s will also be considered).

[22] We begin with a comparison of the viscosity-free and internal viscosity models in the absence of CMB electromagnetic and MICG couplings; the first twelve torsional

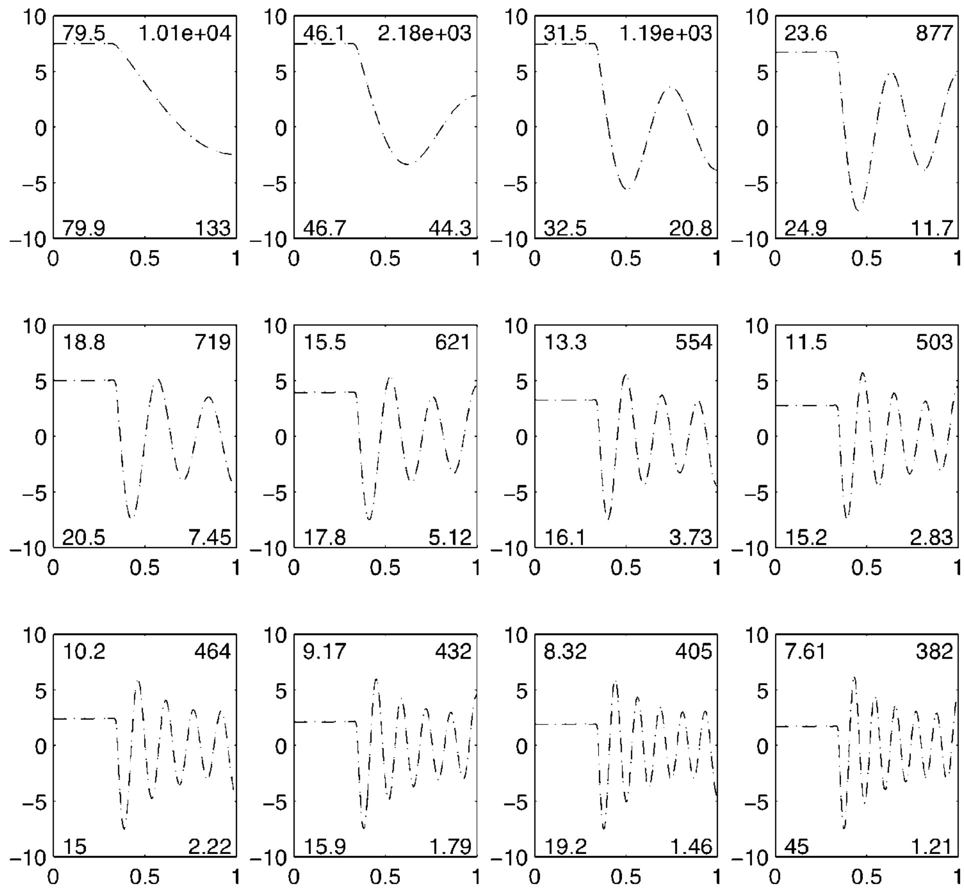

Figure 1. Comparison of the first 12 normal modes of the viscosity-free and internal viscosity models. Note that CMB electromagnetic and MICG couplings are not considered in these models. Normalized angular velocity (u_f) is plotted as a function of normalized radius (s/r_f). Each plot also gives the periods (left-hand corners) and decay times (right-hand corners) for the viscosity-free (ν = 0, dashed lines, top corners) and internal viscosity (ν = 200 m^2/s, dotted lines, bottom corners) cases. The periods and decay times are given in years. For other physical parameters, see Table 1.

oscillation normal modes of both models are plotted in Figure 1. We have set ν = 200 m^2/s in the internal viscosity case to ensure that viscosity plays an important role in that model. As a result, the decay times of the modes in the internal viscosity case have been greatly reduced from the viscosity-free case and the oscillation periods have increased. The effect of viscosity is greater for the higher harmonics. Despite the high-viscosity value assumed, the spatial structure of the normal modes has not been noticeably altered by the inclusion of viscosity in this model, this issue will be dealt with in detail below.

[23] In order to further investigate the effect of viscosity on the temporal behavior of the torsional oscillation normal modes we consider the periods and decay times of the normal modes for a suite of models in which the viscosity is varied (Figures 2a and 2b). As the viscosity of the model is increased the normal modes cross a transition from damped oscillations to pure decay. This transition is seen in the sudden increase of the normal mode periods to infinity. The critical viscosity value at which this transition occurs depends on the period of the normal mode; higher-frequency, shorter-wavelength normal modes cross the transition at a lower-viscosity value. The critical viscosity value for the normal mode with a viscosity-free period of 80 years is $\sim$2000 m^2/s; torsional oscillation normal modes with a viscosity-free period of $\sim$2 years have $\nu_{crit} \simeq$ 55 m^2/s.

[24] In (Figures 2c and 2d) we include the effects of both CMB electromagnetic and MICG coupling. Inclusion of gravitational coupling between the mantle and inner core results in a new normal mode with a period of approximately 6 years (for the assumed value of $\tilde{\Gamma}_g$). This mode does not undergo the same transition to overdamping as the torsional oscillation normal modes. As shown below, the transition occurs when viscous dissipation becomes large compared to the restoring torque associated with shear of the magnetic field between oscillating fluid cylinders. For the MICG normal mode the main restoring torque is instead the gravitational coupling between the mantle and inner core. The MICG mode will undergo a transition to viscous overdamped decay; however, this occurs at a viscosity value above the critical value for torsional oscillation normal modes of a similar period.

[25] The inclusion of electromagnetic coupling at the CMB introduces an additional source of dissipation. Our adopted value for total conductance for the lower mantle, which is consistent with constraints inferred from studies of

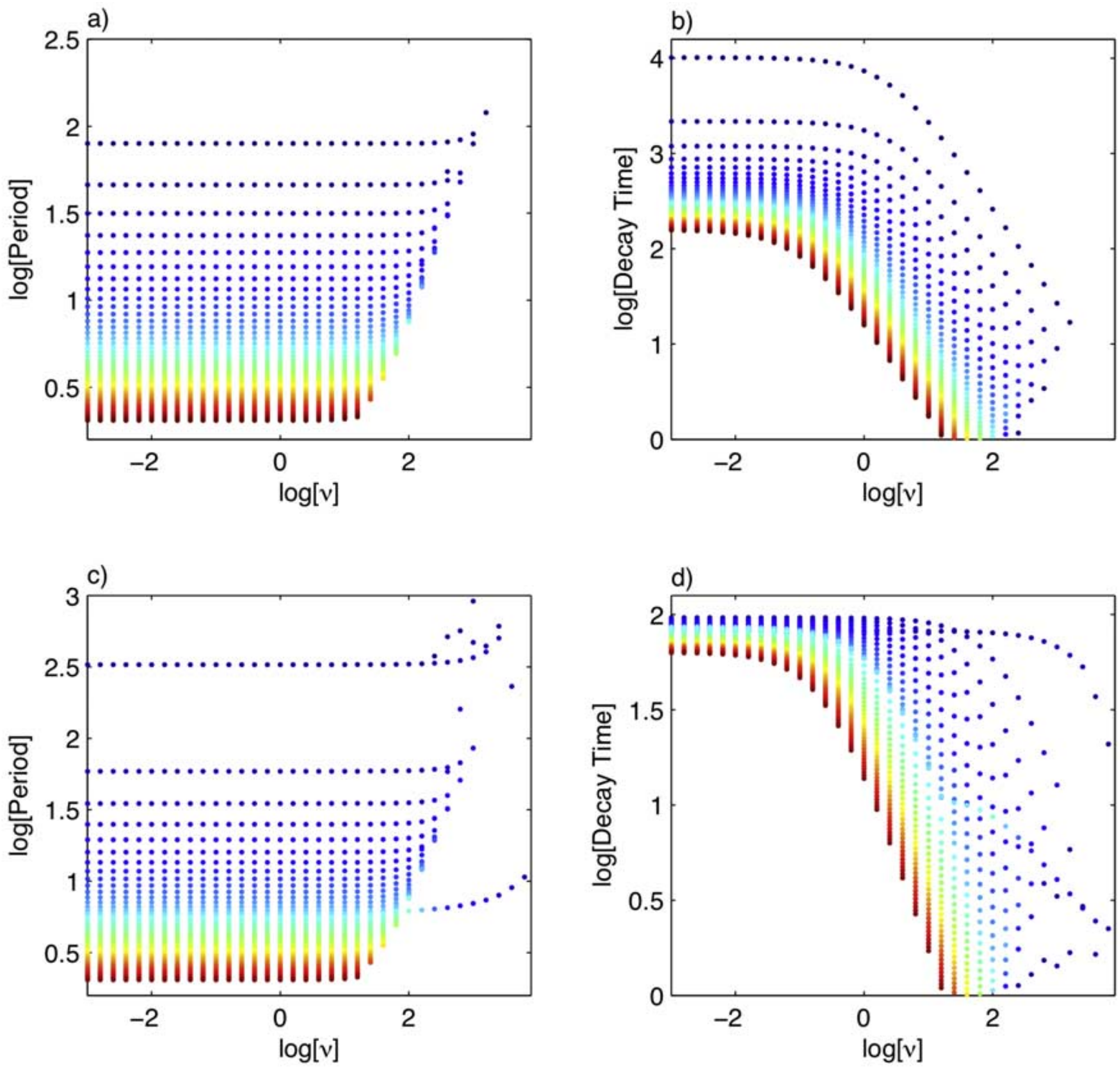

Figure 2. Dependence on viscosity of the torsional oscillation periods and decay times for the internal viscosity model. (a and b) Models that consider only ICB electromagnetic coupling and internal viscous effects. (c and d) Models that also include CMB electromagnetic and MICG couplings. At each value of viscosity, modes are color-coded from longest period (blue) to shortest period (red). Only oscillatory modes with a period greater than 2 years and a decay time greater than 0.25 years are plotted. Physical parameters of the model are given in Table 1.

the Earth's nutations [*Mathews et al.*, 2002], reduces the viscosity-free decay time of the decadal period normal modes by 1–2 orders of magnitude. However, the viscosity value at which the modes transition from oscillatory to overdamped behavior is not greatly altered provided that electromagnetic dissipation at the CMB is not, by itself, sufficient to overdamp the normal modes; this is the case for the value of lower-mantle conductance inferred from nutation studies.

[26] Viscous dissipation within the bulk of the fluid core can alter both the periods and decay times of the torsional oscillation normal modes. The periods of the normal modes are only strongly affected by viscosities close to the critical value; the decay times are more sensitive to the inclusion of viscous dissipation. Whether or not viscous effects are significant in the Earth will depend not only on the viscosity of the fluid core, but also on the relative strengths of other dissipative mechanisms. If electromagnetic coupling at the CMB is as strong as that consider in Figures 2c and 2d a viscosity value of $o\{10^{-2}\}$ m^2/s is required for the additional, viscous, dissipation to produce a 10% reduction in the decay times of the short-period modes. For decadal period torsional oscillation normal modes the viscosity must reach a value of $o\{10^{1}\}$ m^2/s in order to cause a 10% reduction of the decay times. In the absence of electromagnetic dissipation at the CMB (Figures 2a and 2b) a 10% reduction in the decay times of decadal period normal modes requires a viscosity of $o\{10^{-1}\}$ m^2/s.

[27] Allowing a variable B_s field may also change the value at which internal viscous dissipation of the normal modes becomes important; the morphologies that we investigated altered the relevant viscosity value by less than an order of magnitude. Decadal period torsional oscillation normal modes were always less sensitive to the inclusion

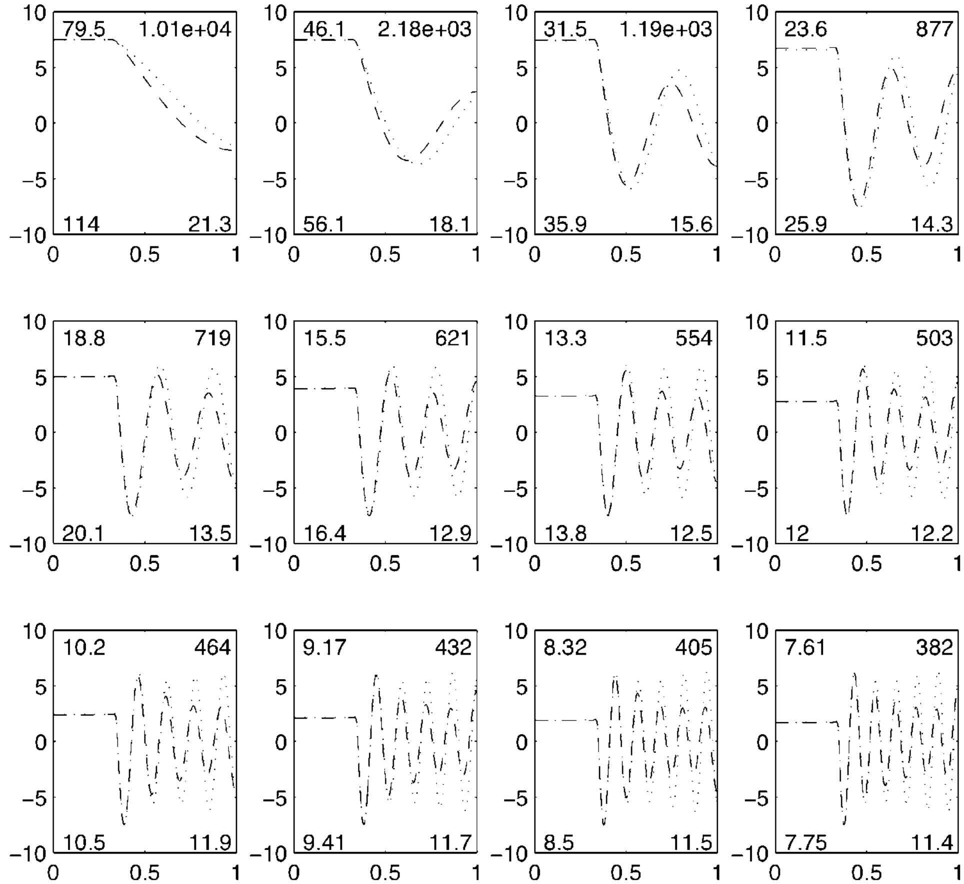

Figure 3. Comparison of the first 12 normal modes of the viscosity-free and full-viscosity models. Note that CMB electromagnetic and MICG couplings are not considered in these models. Normalized angular velocity (u_f) is plotted as a function of normalized radius (s/r_f). Each plot also gives the periods (left-hand corners) and decay times (right-hand corners) for the viscosity-free ($\nu = 0$, dashed lines, top corners) and the full-viscosity ($\nu = 0.5$ m^2/s, dotted lines, bottom corners) cases. The periods and decay times are given in years. For other physical parameters, see Table 1.

of internal viscous dissipation than the short-period modes. Therefore, if viscous dissipation in the bulk of the fluid core is to be an important factor in dissipation of the observed, decadal torsional oscillations of the Earth, then the fluid core must have a viscosity much larger than that estimated from liquid metal studies.

[28] When the effect of viscous boundary layers at the CMB and ICB are included, viscous dissipation becomes more important, especially for the longer-period modes. Figure 3 shows the first twelve torsional oscillation normal modes for the viscosity-free case and for the full-viscosity case with $\nu = 0.5$ m^2/s, CMB electromagnetic and MICG couplings are not included. As in the internal viscosity case, the periods are increased and decay times reduced by the inclusion of viscous effects; however, in the full-viscosity case the lower harmonics are more sensitive to the addition of viscosity. Also in contrast to the previous case, the spatial structure of the normal modes is altered by the inclusion of the viscous boundary layers. The effect of viscosity on the spatial structure of the normal modes is examined in greater detail in the following section.

[29] In Figure 4 we plot the periods and decay times of the normal modes of the full-viscosity model for a range of viscosity values. The transition from oscillatory behavior to pure decay may again be seen in the sudden increase of normal mode periods; however, the long-period and short-period modes undergo separate transitions. For the suite of models that do not include CMB electromagnetic or MICG couplings (Figures 4a and 4b) the critical damping occurs at a viscosity value of $\sim$1 m^2/s for the torsional oscillation normal mode with a viscosity-free period of 80 years; the value of ν_{crit} being higher for slightly shorter period normal modes. The short-period modes reach a separate value of ν_{crit} similar to that of the internal viscosity case; normal modes with viscosity-free periods of $\sim$2 years have $\nu_{crit} \simeq 55$ m^2/s, with slightly longer-period modes having a higher value of ν_{crit}. We again find that the adopted CMB electromagnetic and MICG couplings do not greatly alter the values of ν_{crit} (Figures 4c and 4d). As will be discussed further in the following section, the transition in the temporal behavior of the short-period modes continues to be determined by the effect of internal viscous dissipation, whereas the transition in the behavior of the long-period modes is now controlled by the effect of the Ekman boundary layers.

[30] The effect of viscosity on the temporal behavior of the long-period modes is more important in the full-viscosity

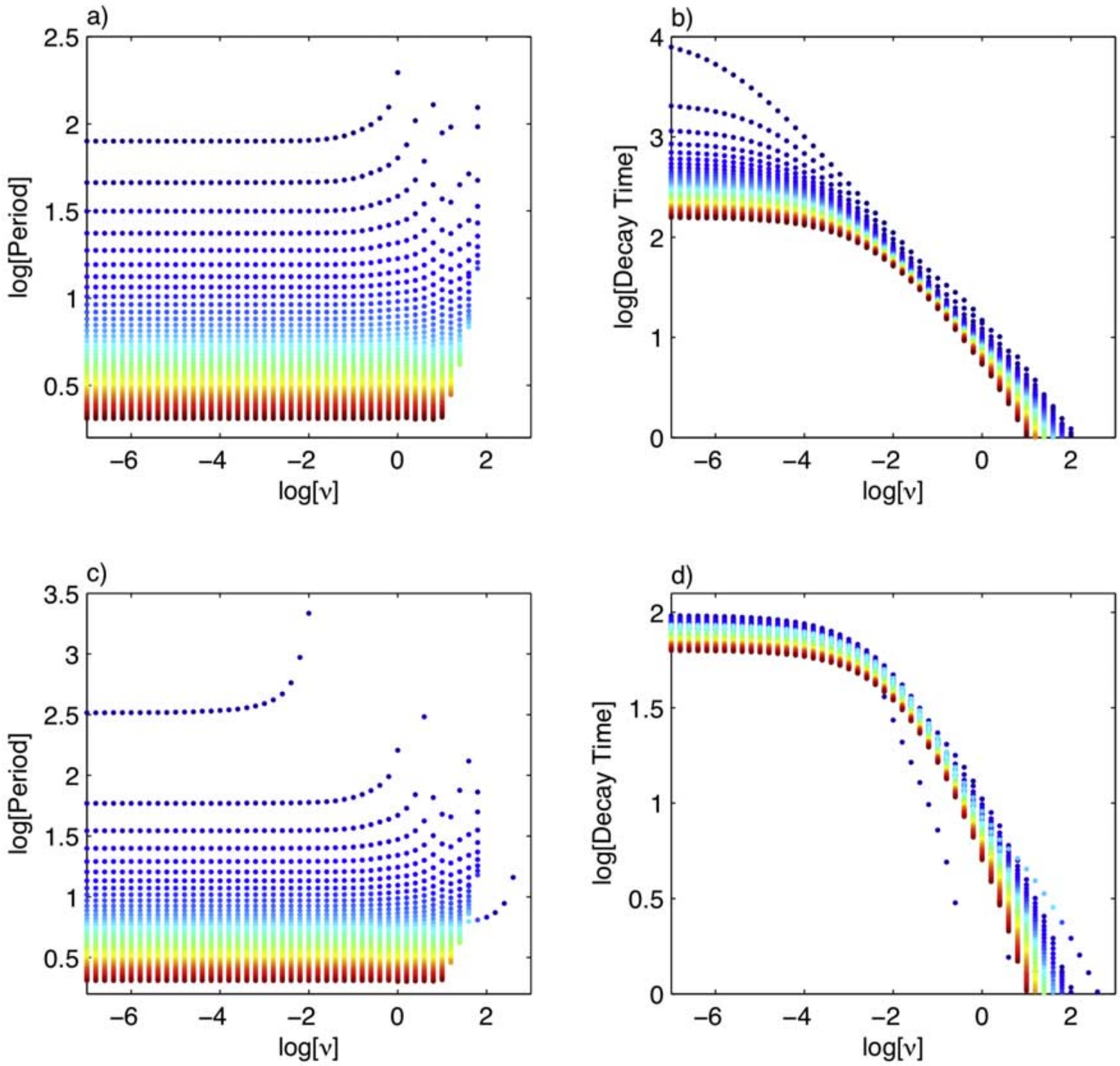

Figure 4. Dependence on viscosity of the torsional oscillation periods and decay times for the full-viscosity model. (a and b) Models that consider only ICB electromagnetic coupling and internal and boundary viscous effects. (c and d) Models that also include CMB electromagnetic and MICG couplings. At each value of viscosity, modes are color-coded from longest period (blue) to shortest period (red). Only oscillatory modes with a period greater than 2 years and a decay time greater than 0.25 years are plotted. Physical parameters of the model are given in Table 1.

model than in the internal viscosity case. In the absence of electromagnetic coupling at the CMB viscosities of $o\{10^{-7} - 10^{-6}\}$ m^2/s are sufficient to reduce the decay times of decadal period normal modes by 10% from their viscosity-free values, with the longer-period modes being the most strongly affected. If CMB electromagnetic coupling is present, then the viscosity must reach a value of $o\{10^{-4}\}$ m^2/s to reduce the normal mode decay times by 10% from their viscosity-free values. As in the internal viscosity case, changes in B_s morphology alter these viscosity values, but by less than an order of magnitude. Regardless of the level of CMB electromagnetic coupling, a viscosity of $o\{10^{-1}\}$ m^2/s is required to produce a 10% change in the periods of decadal normal modes in the full-viscosity models.

[31] Thus viscous, boundary layer, effects could have an important influence on the decay times, but not periods, of the Earth's decadal torsional oscillation normal modes for viscosities as low as those suggest by liquid metal studies, provided that electromagnetic coupling at the CMB is much weaker than that inferred from the study of Earth's nutations. However, in this case the decay times of the decadal normal modes would be in excess of 1000 years; this does not agree with the observationally inferred behavior of the Earth's torsional oscillations. The constraints on CMB electromagnetic coupling obtained from nutation studies can provide dissipation compatible with the inferred damping of torsional oscillations. With that level of CMB electromagnetic coupling viscosity will not significantly affect decadal torsional oscillations, provided that viscosity of the Earth's core is as low as the liquid metal studies suggest.

[32] If the fluid core viscosity is as large as allowed from other dynamic constraints [e.g., *Lumb and Aldridge*, 1991; *Secco*, 1995], then viscous boundary effects could be large

enough to affect the periods of the lowest harmonics and would greatly reduce the decay times of torsional oscillation normal modes, even in the presence of CMB electromagnetic coupling. For example, *Mathews and Guo* [2005] considered a number of different CMB electromagnetic coupling scenarios. For the scenario most similar to the one that we have used they obtained a maximum viscosity of 3.1×10^{-2} m^2/s. Adopting that viscosity increases the period of the longest, decadal mode in Figures 4c and 4d by ~5% and reduces the decay time of that mode by ~65%, relative to the viscosity-free model.

4. Discussion

[33] We now consider in more detail the temporal and spatial structure of the viscously damped torsional oscillation normal modes. In particular, we wish to explain the noted transition from damped oscillations to pure exponential decay, the different values of ν_{crit} at which the transition occurs in the internal viscosity and full-viscosity cases, and the relatively small influence of viscosity on the spatial structure of the normal modes. With an increased understanding of the factors that control the dynamics of the system we then use the model to obtain an upper bound on the viscosity of the fluid core in the vicinity of the CMB.

[34] We begin by considering the internal viscosity case. For mathematical convenience we consider a magnetic field for which B_s is constant and, in order to isolate viscous effects, we ignore electromagnetic couplings at the CMB and ICB. Under these simplifications equation (7) reduces to

$$p(s)\frac{\partial^2 u_f}{\partial t^2} = \frac{\partial}{\partial s}\left(p(s)\left(c^2 + \nu\frac{\partial}{\partial t}\right)\frac{\partial u_f}{\partial s}\right), \tag{11}$$

where $c = \sqrt{B_s^2/\rho\mu}$ is the shear Alfvén wave speed and $p(s) = s^3\,(z_f - z_i)$. This expression can be transformed to two ordinary differential equations via separation of variables. Substituting $u_f(s, t) = \phi(s)T(t)$ leads to

$$T'' + k^2\nu T' + k^2c^2T = 0 \tag{12}$$

$$(p\phi')' + k^2p\phi = 0, \tag{13}$$

where k is an undetermined constant and the primes indicate differentiation with respect to s or t as appropriate.

[35] Equation (13) is an eigenvalue problem that determines the spatial structure of the torsional oscillation normal modes as well as the eigenvalues, k. Note that equation (13) is only strictly valid on the open interval $s = (0,r_f)$ because $p(s)$ is zero at the endpoints. However, solutions for $\phi(s)$ can be extended arbitrarily close to the points $s = 0,r_f$ and the boundary conditions applied in the corresponding limit. The temporal behavior of the normal modes is given by the damped oscillator equation (12); this equation reduces to simple harmonic motion if viscosity is neglected, and it reduces to pure exponential decay if the magnetic field is zero.

[36] Although the temporal behavior of the normal modes depends on the magnitude of physical parameters such as viscosity and magnetic field strength (equation 12), the eigenvalues and the spatial structure of the normal modes of the internal viscosity model do not; equation (13) depends only on the core geometry and the spatial variability of physical parameters. Thus identical magnetic field morphologies would correspond to suites of torsional oscillation normal modes with the same eigenvalues, k, and spatial structures, ϕ. Similarly, introducing a constant internal viscosity to this constant B_s model alters the temporal behavior, but not the eigenvalues and spatial structures of the normal modes, even if the critical damping transition is crossed. This insensitivity of the constant B_s model to the inclusion of viscosity is seen in Figure 1. If B_s varied considerably throughout the core, then the inclusion of a large viscosity could significantly alter the shape of the torsional oscillation normal modes.

[37] We now consider the temporal behavior of the internal viscosity model's normal modes in more detail. The general solution to equation (12) is

$$T(t) = a_1e^{r_1t} + a_2e^{r_2t}, \tag{14}$$

and the roots of the associated characteristic equation are

$$r_{1,2} = -\frac{1}{2}k^2\nu \pm \frac{1}{2}\sqrt{k^4\nu^2 - 4k^2c^2}. \tag{15}$$

If the viscosity is zero, then the solution reduces to the expected simple harmonic oscillator,

$$T(t) = a_1e^{i\omega_0t} + a_2e^{-i\omega_0t}, \tag{16}$$

where $\omega_0 = kc$. Thus the eigenvalues of equation (13) are the wave numbers of the viscosity-free normal modes. The inclusion of a small internal viscosity results in an underdamped solution

$$T(t) = e^{-\frac{1}{2}k^2\nu t}\left(a_1e^{i\omega_1t} + a_2e^{-i\omega_1t}\right), \tag{17}$$

which includes both an exponentially decaying term as well as a shift to a lower, effective frequency $\omega_1 = \sqrt{\omega_0^2 - \frac{1}{4}k^4\nu^2}$. As the viscosity is increased the decay time shortens and the period lengthens until the critically damped case is reached. Since only internal viscous effects are considered, the viscous damping scales as $o\{E\}$ and critical damping is reached when $(2kc)/(k^2\nu_{crit}) = 1$; that is, when the viscous diffusion timescale becomes sufficiently small with respect to the period of the oscillation. At or above the critical viscosity value, viscous damping dominates and the solution no longer admits oscillatory behavior. For sufficiently large viscosity the magnetic field becomes negligible and the pure decay case is recovered,

$$T(t) = a_1 + a_2e^{-k^2\nu t}. \tag{18}$$

[38] Since the eigenvalues, k, are the wave numbers of the normal modes in the absence of viscosity, we can use the frequencies of the viscosity-free normal modes to estimate the viscosity value at which the critical damping condition will be reached in the internal viscosity model. This value

will be inversely proportional to the undamped frequency of the given normal mode; we expect $\nu_{crit} \approx \frac{2c^2}{\omega_0}$. For B_s equal to a constant value of 0.2 mT, the critical value of viscosity would range from $o\{10^1 - 10^3\}$ m^2/s for normal modes with viscosity-free periods of the order of 1–100 years. This estimate agrees well with the results shown in Figure 2.

[39] We consider now the full-viscosity case, which includes the effect of the Ekman boundary layers. In order to more easily investigate the influence of the Ekman layers, we again consider a simplified core model to obtain an order of magnitude estimate of the importance of viscous core-mantle coupling; we assume that B_s is constant and that there is no electromagnetic coupling between the fluid core and surrounding solid regions. We further consider the core to be completely fluid and cylindrical; as shown in Appendix A this approximation is valid for the present purpose of determining a first-order approximation of the temporal and spatial structure of the normal modes. Additionally, we assume that the angular velocity of the mantle is sufficiently small that it can be neglected with respect to the angular velocity of the fluid in equation (6); due to the mantle's relatively large moment of inertia, and the fact that the mantle rotates as a solid body whereas there is differential rotation between regions of the fluid core, typical solutions have u_m 2 orders of magnitude smaller than the average value of u_f.

[40] Under these assumptions we again use separation of variables and obtain

$$T'' + \left(\left(\frac{\nu\Omega}{r_f^2}\right)^{1/2} + k^2\nu\right)T' + k^2c^2T = 0 \tag{19}$$

$$(\tilde{p}\phi')' + k^2\tilde{p}\phi = 0, \tag{20}$$

where $\tilde{p} = s^3$. For this simplified version of the full-viscosity model the inclusion of the Ekman boundary layers does not affect the eigenvalues and spatial structure of the normal modes. The assumptions we have employed cause the spatial dependence of the viscous coupling at the CMB to be the same as that of the magnetic tension in the bulk of the fluid core. In general, the spatial dependence of the viscous coupling will not have the same spatial variation as the magnetic tension, and the inclusion of viscous boundary layers will affect the spatial structure of the normal modes (recall Figure 3).

[41] The temporal behavior of the system is again a damped oscillation (equation (19)) and the solution for the roots of the associated characteristic equation gives the condition for critical damping

$$\left(\frac{\Omega\nu_{crit}}{r_f^2}\right)^{1/2} + k^2\nu_{crit} = 2kc. \tag{21}$$

As seen in Figure 4 this condition gives rise to two classes of modes that undergo separate transitions from damped oscillations to pure decay.

[42] For sufficiently large k, i.e., short-wavelength and high-frequency normal modes, the first term on the left-hand side of equation (21) can be neglected, this leads to the same critical condition as in the internal viscosity case (i.e., a balance between the period of the oscillation and the viscous diffusion timescale). For this class of modes the magnetic tension is overcome by the internal viscosity, not by the influence of the viscous boundary layers, and the viscous effects again scale as $o\{E\}$. Torsional oscillation normal modes with periods of the order of 1 year fall in this class and, as above, will become critically damped at a viscosity of $o\{10^1\}$ m^2/s.

[43] For small k, i.e., long-wavelength and low-frequency normal modes, the second term on the left-hand side of equation (21) can be neglected, and the critical condition becomes a balance between the period of the oscillation and the viscous spin-up time for a rapidly rotating system [*Greenspan and Howard*, 1963]. For this class of modes we expect critical damping to occur at $\nu_{crit} \approx (4\omega_0^2 r_f^2)/\Omega$, which corresponds to a critical viscosity of $o\{10^0\}$ m^2/s for modes with viscosity-free periods of the order of 100 years. Thus, in models which include viscous dissipation at the boundaries of the fluid core, viscous effects on long-period modes scale as $o\{\sqrt{E}\}$ leading to a much reduced value of ν_{crit} in comparison to the free-slip case. Furthermore, in the no-slip case the long-period modes have a value of ν_{crit} that is lower than that of the short-period modes; the opposite occurs in models with free-slip boundaries.

[44] The preceding analysis indicates that dissipation in Ekman boundary layers is the most important viscous effect for the Earth's decadal torsional oscillation normal modes. Although electromagnetic coupling at the CMB could produce significant dissipation of decadal torsional oscillations in the Earth we consider here a suite of full-viscosity models that do not include electromagnetic coupling at the CMB (electromagnetic coupling at the ICB is retained). By ignoring electromagnetic friction at the CMB we can obtain an upper bound on the viscosity of the fluid core. This constraint applies mainly to the fluid viscosity near the top of the core, viscous dissipation in the CMB Ekman layer being the mechanism that dominates the damping of decadal torsional oscillation normal modes in this set of models. The relatively small velocity difference between the fluid and solid cores limits the level of dissipation because of coupling at the ICB, recall the low-viscosity limit of Figure 4.

[45] Figure 5 displays the decay times of decadal period torsional oscillation normal modes for a suite of full-viscosity models that lack CMB electromagnetic coupling. Only normal modes with periods between 25 and 100 years are plotted; this period range does not sample the same harmonics at all viscosities because of the change in normal mode periods as the value of viscosity is varied. In Figure 5 we plot results for models in which B_s is constant, and for models in which B_s decreases as $s \rightarrow r_f$. At any given viscosity, the decay times in the variable B_s models are shorter than in the constant B_s model. The normal modes in the variable B_s models have regions of relatively high fluid velocity near the equator of the fluid core, which leads to more viscous dissipation than for the corresponding, constant B_s modes.

[46] *Zatman and Bloxham* [1997, 1998] were able to distinguish two decadal period waves in models of core flow, the decay times of these waves being 74.1 and 95.7 years. Our goal here is not to exactly match these

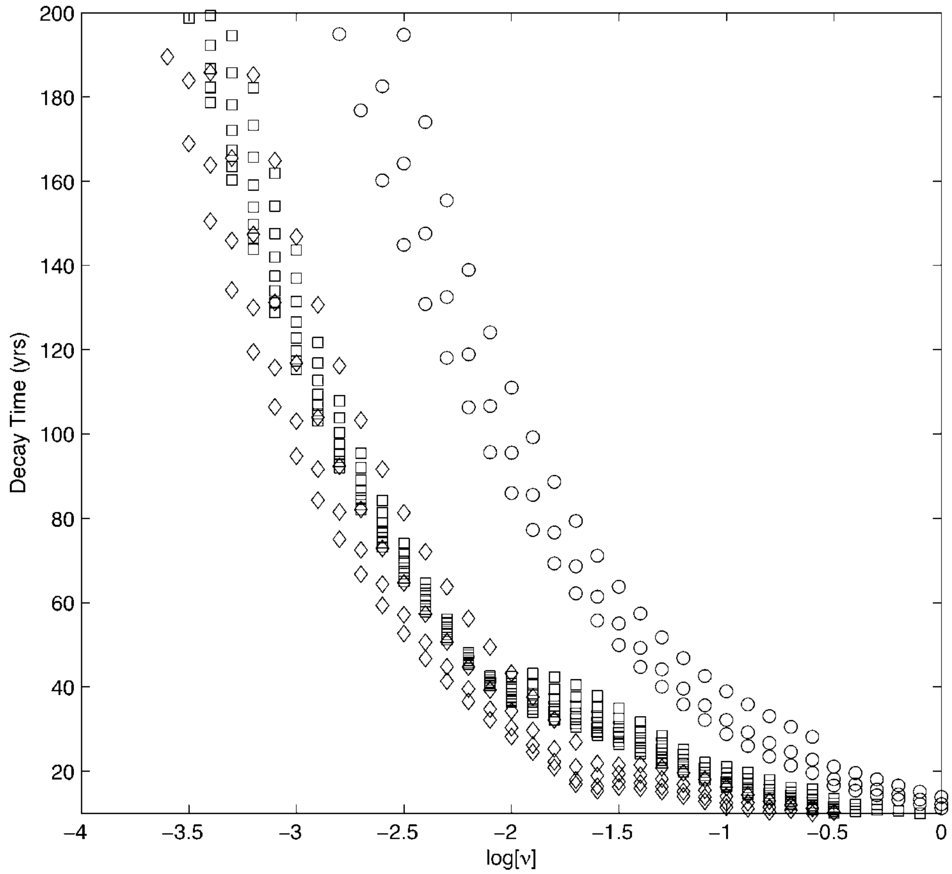

Figure 5. Decay time of decadal torsional oscillations when viscous coupling at the CMB is the primary mechanism of dissipation. The normal modes are calculated for models that have a B_s field that is constant (circles), linearly decreasing outside the tangent cylinder to 10% of its peak value (squares), or based on a dipole throughout the core (diamonds). Only normal modes with periods in the range 25–100 years are plotted.

observed waves, but rather to obtain an order of magnitude bound on the fluid core viscosity. We assume that the decaying waves detected by *Zatman and Bloxham* [1997, 1998] do represent underdamped torsional oscillations and take as our constraint that there be torsional oscillation normal modes with decadal periods and decay times of at least 50 years, in general accordance with the observations. This constraint leads to an upper bound on the fluid core viscosity near the CMB of $v_{max} \approx 4 \times 10^{-2}$ m^2/s, based on our constant B_s model, or $v_{max} \approx 5 \times 10^{-3}$ m^2/s, based on either of our variable B_s models (requiring that the decay times be at least 74.1 years lowers these bounds by about a factor of 2). This bound is based on the dissipation of torsional oscillations, another constraint arises from the observed decadal variations in LOD which require that torsional oscillations produce a net torque on the mantle of $o\{10^{18}\}$ N m. Substitution of a viscosity of $o\{10^{-2}\}$ m^2/s in equation (6), which must then be integrated over the CMB, leads to an estimate of for the net viscous torque on the mantle that is of the required magnitude.

[47] A viscosity of $o\{10^{-2}\}$ m^2/s is consistent with both the observed dissipation of torsional oscillations and the observed LOD variations. This constraint on the viscosity of the fluid core is approximately equal to those recently obtained from VLBI observations of the Earth's nutations [*Mathews and Guo*, 2005; *Palmer and Smylie*, 2005; *Deleplace and Cardin*, 2006]. Our bound is several orders of magnitude above the viscosity values preferred from theoretical or laboratory studies of liquid metals but below the majority of estimates based on geophysical observations [*Lumb and Aldridge*, 1991; *Secco*, 1995] as well as some values suggested by theoretical work on the increased dissipation that arises because of turbulence in core flows [*Braginsky and Meytlis*, 1990].

[48] It is important to note that the obtained values for v_{max} correspond to a fluid core in which $P_m < 1$. Therefore the existence of a high-conductivity layer at the base of the mantle would result in decadal torsional oscillations that are primarily damped by the effects of magnetic diffusion at the CMB. Although not required to explain the observations, it is possible that the dissipation of the Earth's torsional oscillations is entirely due to electromagnetic friction; therefore it is not possible to use this model to obtain a lower bound on the fluid viscosity.

5. Conclusions

[49] We have investigated how viscous effects could alter the spatial and temporal structure of torsional oscillation normal modes. Applying separation of variables to a

simplified model of torsional oscillations showed that the eigenvalues and spatial structures of normal modes of the system are determined primarily by the morphology of the radial magnetic field within the fluid core. In the absence of magnetic diffusion the temporal behavior of the normal modes is determined by the relative magnitudes of the radial magnetic field and the fluid viscosity. If the viscosity is sufficiently large, then the normal modes undergo pure exponential decay rather than damped oscillations. The pseudoperiod of the damped oscillations approaches infinity as the viscosity increases toward a critical value.

[50] For free-slip boundaries the critical viscosity value represents a balance between the period of the oscillation and the viscous diffusion timescale of the motion. With Earth-like values for other physical parameters the transition to pure decay in the internal viscosity model occurs at viscosity values of $o\{10^1 - 10^3\}$ m^2/s for torsional oscillation normal modes having viscosity-free periods of the order of 1–100 years. When the effects of viscous boundary layers are included the critical viscosity value for long-wavelength torsional oscillation normal modes is set by a balance between the period and the viscous spin-up time of a rapidly rotating fluid; the critical viscosity for short-wavelength torsional oscillation normal modes is still determined by the internal viscous effects. In the full-viscosity model the transition to pure decay for normal modes with viscosity-free periods of the order of 100 years occurs at $\nu_{crit} = o\{10^0\}$ m^2/s. To ensure that decadal period torsional oscillation normal modes are below the critical damping threshold in a numerical geodynamo model it would be necessary to consider cases with an Ekman number of at most $o\{10^{-7}\}$ if free-slip boundary conditions are imposed; if no-slip boundaries are used then an Ekman number of at most $o\{10^{-10}\}$ would be required. Provided that other parameters, such as magnetic diffusivity, are properly accounted for numerical models may be able to reproduce the approximate spatial structures and periods (although not the decay times) of torsional oscillation normal modes at viscosity values several orders of magnitude above that expected for the Earth's core.

[51] At low-viscosity values viscous dissipation would more strongly affect the decay times, as opposed to the periods, of the normal modes. Whether or not viscous effects are important in the Earth depends on both the viscosity of the fluid core and on the level of electromagnetic friction at the fluid core boundaries. Even in the absence of CMB electromagnetic dissipation, a viscosity of $o\{10^{-1}\}$ m^2/s is required for viscous dissipation in the bulk of the fluid core to produce a 10% reduction in the decay times of decadal period normal modes. This value is greater than the viscosity constraint that we have derived. Therefore internal viscous dissipation is unlikely to be important in the Earth's decadal torsional oscillations. It may be necessary to include viscous boundary effects in torsional oscillation studies, depending on the values of $\sigma_m\Delta$ and ν. If the conductance of the lowermost mantle is equal to that inferred from nutation studies, then a viscosity of $o\{10^{-4}\}$ m^2/s is required for viscous boundary effects to produce a 10% reduction in the decay times of decadal torsional oscillation normal modes. This value of ν is 2 orders of magnitude below the constraint we have derived, but 2 orders of magnitude above that suggested by studies of liquid metals. Viscous boundary effects would be important even for viscosities as low as $o\{10^{-6}\}$ m^2/s if electromagnetic CMB coupling is much lower than inferred from the study of the Earth's nutations.

[52] Decadal changes in LOD are interpreted as the signature of torsional oscillations with decadal periods [e.g., *Jault et al.*, 1988; *Jackson et al.*, 1993]. We find that the existence of oscillatory, decadal period, torsional oscillation normal modes in the Earth's core requires a fluid viscosity less than $o\{10^0\}$ m^2/s. The observationally inferred decay time of decadal torsional oscillation normal modes imposes an upper bound of $o\{10^{-2}\}$ m^2/s for the fluid viscosity in the vicinity of the CMB. This value is compatible with the torque required to produce observed decadal variations in LOD and is comparable to that found by recent nutation studies [*Mathews and Guo*, 2005; *Palmer and Smylie*, 2005; *Deleplace and Cardin*, 2006]. These constraints are below most other observationally derived bounds on viscosity and appear to rule out some of the extreme turbulent viscosity values that have been proposed [e.g., *Braginsky and Meytlis*, 1990].

Appendix A: Effect of Core Geometry Simplifications

[53] In this appendix we investigate the effect that the applied simplifications to core geometry have had on the spatial and temporal structure of the torsional oscillation normal modes. We consider geometries with and without a solid inner core, as well as a cylindrical core geometry which allows an analytic solution to the eigenvalue problem. The governing equation for the spatial variability of the torsional oscillation normal modes under the approximation of a completely fluid, cylindrical core (equation (20) in the main body of the text) is

$$s^2\phi'' + 3s\phi' + k^2s^2\phi = 0. \tag{A1}$$

This is a modified Bessel equation having the solution [e.g., *McQuarrie*, 2003]

$$\phi(s) = s^{-1}(b_1J_1(ks) + b_2Y_1(ks)), \tag{A2}$$

where J_1 and Y_1 are the first-order Bessel functions of the first and second kind, respectively.

[54] Regularity conditions on the angular velocity require that $\phi' = 0$ at $s = 0$, r_f. As noted in the main body, the derived equations for the spatial structures of the normal modes do not apply at the limits $s = 0$, r_f; however, solutions for $\phi(s)$ can be extended arbitrarily close to the boundaries and the regularity conditions applied in the corresponding limits. The boundary condition at $s = 0$ requires that $b_2 = 0$. The boundary condition at $s = r_f$ can then be used to determine the eigenvalues of the torsional oscillation system. These results, along with the solutions to equations (12) or (19), describe the torsional oscillation normal modes of a completely fluid, cylindrical core to within an arbitrary multiplicative constant.

[55] In Figure A1 we compare the analytic results obtained from equation (A2) for the first four normal modes

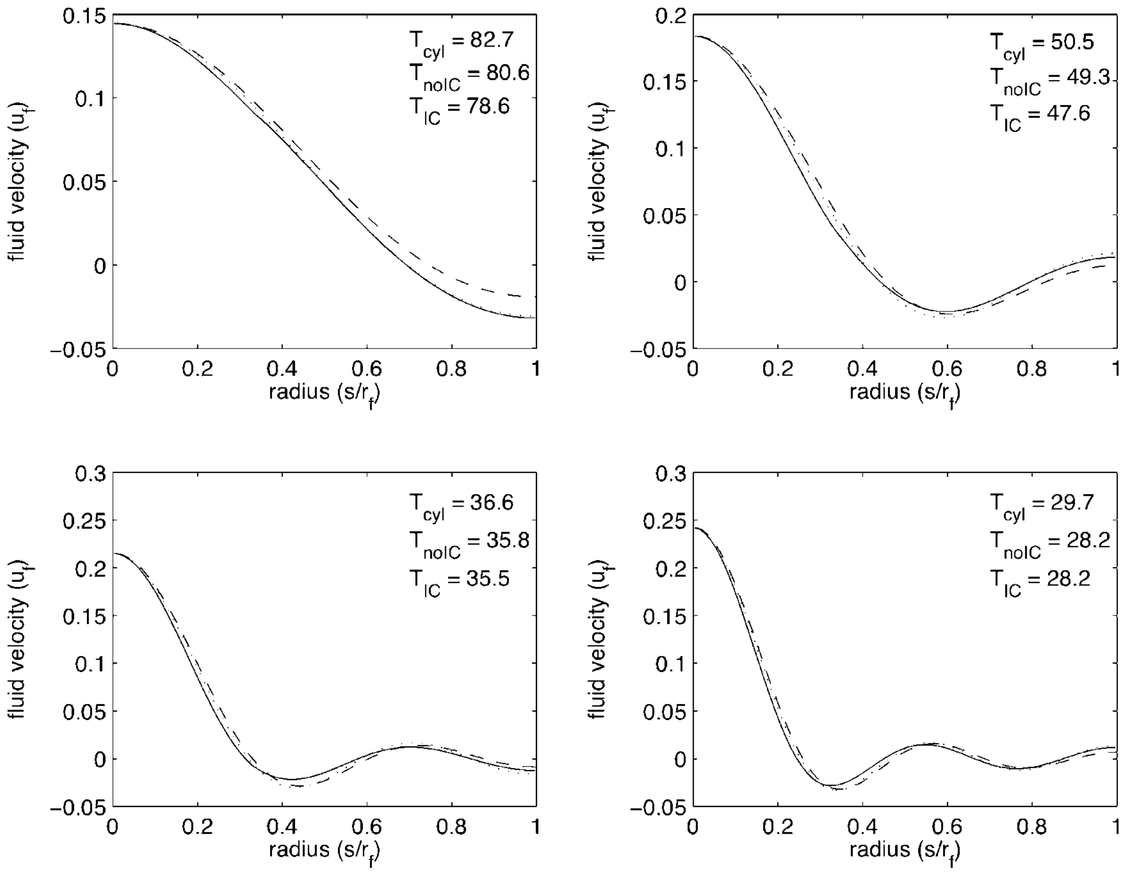

Figure A1. Comparison of the spatial structure and periods of the first four normal modes as determined by the analytic solution for a completely fluid, cylindrical core (dashed line, T_{cyl}) and the numerical solutions for a completely fluid, spherical core (dotted line, T_{noIC}) and a spherical core with a solid inner core (solid line, T_{IC}). Periods are given in years. All models assume constant B_s = 0.2 mT and no coupling between the fluid core and the surrounding solid regions.

of a viscosity-free, completely fluid, cylindrical core, with the corresponding modes determined numerically for two other viscosity-free models. In one case the entire spherical core is assumed to be fluid, and in the other the solid inner core is included. None of the models considered in Figure A1 consider coupling between the fluid core and the surrounding solid regions. Figure A1 shows that the periods and spatial structures to the normal modes are not greatly affected by the geometry of the core model, the similarity between the cylindrical core and the more realistic geometries improving at higher frequencies. The relative unimportance of core geometry to short-period torsional oscillation normal modes has been shown previously [*Mound and Buffett*, 2003].

[56] It is interesting to note the importance of the conductive solid core on the behavior of the modes near $s = 0$ (compare Figures 2 and A1). The models of Figure A1 have either no inner core, or an inner core that is uncoupled to the fluid; in the Earth, the highly conductive inner core couples strongly to the fluid through electromagnetic friction producing nearly rigid body rotation of the fluid within the tangent cylinder [*Braginsky*, 1970, 1984; *Buffett*, 1998; *Mound and Buffett*, 2003]. In the absence of coupling to the solid core the strong radial dependence of the fluid moment density (equation (2)) becomes obvious, with cylinders near the rotation axis obtaining much higher angular velocities than cylinders in the bulk of the fluid core. Electromagnetic coupling to the conductive solid core prevents such rapid fluctuations in the Earth at present; however, strong toroidal motions near the rotation axis might be expected in an entirely fluid core, such as that of the early Earth. Coupling between the fluid and solid cores also alters the period of the normal modes; the approximations employed here lead to normal modes with periods that are longer than the equivalent normal modes in models with electromagnetic ICB coupling. The difference between the periods of the normal modes for these approximate solutions and the periods of the equivalent modes in models that include coupling to the inner core is never greater than ~45%; therefore the approximate solution can be used to obtain first-order estimates of the behavior of torsional oscillation normal modes.

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B. Buffett, Department of Geophysical Sciences, University of Chicago, Chicago, IL 60637, USA. (buffett@geosci.uchicago.edu)

J. Mound, School of Earth and Environment, University of Leeds, Leeds LS2 9JT, UK. (j.mound@see.leeds.ac.uk)