
A generalization of the fractal/facies model

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Abstract In order to generalize the fractal/facies concept, a new stochastic fractal model for $\ln(K)$ increment probability density functions (PDFs) is presented that produces non-Gaussian behavior at smaller lags and converges to Gaussian at larger lags. The model is based on the classical Laplace PDF. The new stochastic fractal family is called fractional Laplace motion (fLam) having stationary increments called fractional Laplace noise (fLan). This fractal is different from other fractals because the character of the underlying increment PDFs changes dramatically with lag size, which leads to lack of self-similarity. Data also appear to display this characteristic. In the larger lag size ranges, approximate self-affinity does hold. The basic field procedure for further testing of the fractional Laplace theory is to measure $\ln(K)$ increment distributions along transects, calculate frequency distributions from the data, and compare results to appropriate fLan family members. The variances of the frequency distributions should also change with lag size (scale) in a prescribed manner. There are mathematical reasons such as the geometric central limit theorem, for surmising that fLam/fLan may be more fundamental than other

approaches that have been proposed for modeling $\ln(K)$ frequency distributions.

Résumé Afin de généraliser le concept fractale/faciès, un nouveau modèle fractal stochastique est présenté pour les fonctions de densité de probabilité (FDP) d'accroissement de $\ln(K)$; il présente des comportements non-gaussiens pour des petits décalages et converge vers le gaussien pour des décalages plus grands. Le modèle est basé sur la FDP classique de Laplace. La nouvelle famille fractale stochastique est appelée mouvement fractionnaire laplacien (fLam) et présente des accroissements stationnaires appelés bruit fractionnaire laplacien (fLan). Ce fractal est différent d'autres fractals car le caractère des FDP d'accroissement sous-jacentes change considérablement avec la grandeur du décalage, ce qui conduit à un manque d'auto-similarité. Les données semblent également présenter cette caractéristique. Pour la gamme des décalages plus grands, une auto-affinité approximative a lieu. La procédure de base pour des tests supplémentaires sur la théorie du fractionnaire laplacien consiste à mesurer les distributions d'accroissement de $\ln(K)$ le long de coupes transversales, à calculer les distributions de fréquence des données et à comparer les résultats aux membres de famille de fLan appropriés. Les variances des distributions de fréquence devraient également changer avec la grandeur du décalage (échelle) suivant une méthode imposée. Il y a des raisons mathématiques, tel que le théorème géométrique de la limite centrale, qui permettent de soupçonner que l'approche fLam/fLan puisse être plus fondamentale que d'autres approches qui ont été proposées pour modéliser les distributions de fréquence de $\ln(K)$.

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Resumen Con objeto de generalizar el concepto de facies/fractales se presenta un nuevo modelo de fractales estocásticos para funciones de densidad de probabilidad (FDPs) del incremento $\ln(K)$ el cual produce un comportamiento no-Gaussiano en intervalos pequeños y converge en intervalos mayores. El modelo se basa en la clásica FDP Laplace. La nueva familia de fractales estocásticos se denomina señal de Laplace fraccionaria (fLam) teniendo incrementos estacionales denominados ruido de Laplace fraccionario (fLan). Este fractal es diferente de otros fractales porque el tipo de cambio del incremento subyacente en la FDP ocurre dramáticamente

con el tamaño del intervalo, lo cual conduce a falta de auto-similaridad. Los datos también parecen mostrar esta característica. En los rangos con tamaño del intervalo más grandes se manifiesta auto-afinidad aproximada. El procedimiento básico de campo para evaluación posterior de la teoría de Laplace fraccionada es medir las distribuciones del incremento de $\ln(K)$ a lo largo de cortes transversales, calcular distribuciones de frecuencia para los datos, y comparar resultados para miembros de familia fLan apropiados. Las varianzas de las distribuciones de frecuencia también deberían de cambiar con el tamaño del intervalo (escala) de una manera prescrita. Existen razones matemáticas, tal como el teorema del límite central geométrico, para suponer que fLam/fLan puede ser más fundamental que otros enfoques que han sido propuestos para modelizar las distribuciones de frecuencia de $\ln(K)$.

Keywords Facies · Fractal model · Heterogeneity · Groundwater hydraulics · Geostatistics

Introduction

During recent years, several different stochastic fractal models such as Lu et al. (2002), have been proposed for representing $\log(K)$ and $\log(K)$ increment structure in heterogeneous sediments, where $\log(K)$ is the natural logarithm of hydraulic conductivity in this communication; data may be analyzed using any log base because logs using different bases are constant multiples of each other. Increments are defined as $\log[K(x+h)] - \log[K(x)]$, where h is the lag, or separation, between measurements, and x specifies a series of equally-spaced measurement points. Both $\log(K)$ and $\log(K)$ increments are mentioned because stochastic fractals are derived from the statistical theory of non-stationary stochastic processes with stationary increments (Molz et al. 2004). Prominent among these processes have been models based on the Gaussian or Lévy-stable probability density functions (PDFs; Molz et al. 2004). Using K measurements and estimates from four different facies of an alluvial fan (floodplain, levee, channel, and debris-flow deposits), Lu et al. 2002 were able to show that mixing purely Gaussian PDFs for the $\log(K)$ increments in each facies could lead to the erroneous conclusion that, overall, the sediments-displayed Lévy-stable PDF statistics. With clear reservations, because of undesirable statistical properties such as infinite variance, the Lévy-stable PDF was proposed as a candidate distribution for $\log(K)$ increments (Painter 1996). One implicit implication of the Lu et al. (2002) work was that $\log(K)$ increment distributions within a single, well-defined facies would be Gaussian. Indeed, there was evidence that for larger lags, h , the $\log(K)$ increment PDFs would approach Gaussian (Liu and Molz 1997). However, there was solid evidence also for non-Gaussian behavior of the $\log(K)$ increments for smaller lags (Painter 1996, 2001), with the PDFs involved also being non-Lévy stable.

The situation may be further clarified by the development of a new stochastic fractal based not on Lévy-stable

or Gaussian statistics, but on statistics of the Laplace PDF (also known as the double-exponential) and its generalizations (Kotz et al. 2001; Meerschaert et al. 2004; Kozubowski et al. 2006). Therefore, the main objective of this technical note is to describe how this new fractal, called fractional Laplace motion, compares to well-known fractals and can be used to generalize the fractal/facies model so that it is more compatible with measurements over a wider range of scales (lag sizes).

Data

The new stochastic fractal, fractional Laplace motion, will be introduced and explained within the context of an intrinsic permeability (k) data set obtained at a field site in Utah, USA, using the small drill-hole version of the gas mini-permeameter (Dinwiddie et al. 2003). A preliminary analysis of these data in the form of a cumulative distribution function (CDF) was presented in Fig. 4 of Lu et al. (2002), where it was concluded that the CDF was well-fit by a Gaussian CDF. A more detailed analysis of the same data set presented in Fig. 11 of Castle et al. (2004) in the form of a more discerning PDF fit, and recreated herein as Fig. 1, showed a distinct peaking around the mean. This peaking raised questions concerning the preliminary conclusion that the $\log(k)$ increment data exhibited Gaussian behavior. The tail behavior appeared to fit a Gaussian PDF well, but the peaking in the vicinity of the mean, where most of the data are centered and expected to have the least statistical “noise”, led to a Chi Square test supporting a Gaussian fit at only the 51% confidence level (Castle et al. 2004).

To further confirm the non-Gaussian nature of the data, statistical analysis of kurtosis was performed. The kurtosis for the normal distribution is always equal to 3, regardless of the model parameters (the mean and the standard deviation). The estimated value of kurtosis for these data is 3.66, which exceeds the value for a normal distribution. A standard test for kurtosis (Anscombe and Glynn 1983) yields the p -value

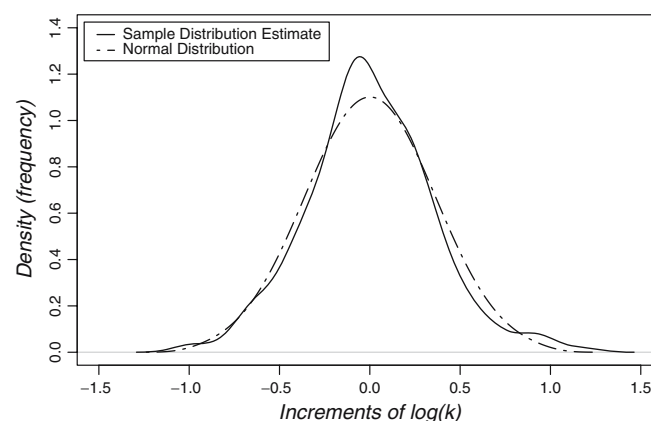


Fig. 1 Frequency distribution for the $\log(k)$ increments (k = intrinsic permeability) shown in Fig. 3a,b along with the best-fit Gaussian PDF. The increments resulted from the $\log(k)$ data shown in Fig. 2a,b

of 0.02417, so the null hypothesis that these data are Gaussian is rejected at the 95% significance level, confirming the non-Gaussian character of the data. This result, along with similar non-Gaussian properties observed in other $\log(k)$ and $\log(K)$ increment data sets (Painter 2001; Meerschaert et al. 2004), led to the present re-examination of the Utah data using the new fractional Laplace model.

Plots of the data utilized in the present study (269 measurements) are shown in Fig. 2a,b. The data were obtained along two nominally horizontal (along bedding) transects on a near-vertical outcrop face of a bioturbated, lower-shoreface sandstone (Castle et al. 2004). The measurements were separated horizontally by a distance (lag) of 15 cm. At the 15 cm lag, the $\log(k)$ increments are shown in Fig. 3a,b. This plot is assumed to represent a so-called “spatially stationary stochastic process” (statistical parameters such as mean, variance, etc., independent of position and, therefore, well-defined) and serves as the basis for analysis. The empirical increment PDF shown in Fig. 1, with a mean of -0.0013 and a standard deviation of 0.350, resulted from analysis of these data.

Gaussian and Lévy-stable stochastic fractals

The name fractional Laplace motion (fLam) was selected in analogy with the well-known Gaussian stochastic

fractal called fractional Brownian motion (fBm). This fractal has been used for some time to represent $\log(k)$ distributions (Molz and Boman 1993; Molz et al. 1997; Meerschaert et al. 2004). Using the terminology of mathematical statistics, both fBm and fLam are non-stationary with stationary increments called fractional Laplace noise (fLan) and fractional Gaussian noise (fGn). By definition, any distribution may be conceived as being the sum of its increments. Therefore, running sums of fGn yields fBm and summing fLan yields the corresponding fLam. Thus, summing the respective transect increments in Fig. 3a,b would yield the corresponding $\log(k)$ distributions shown in Fig. 2a,b, regardless of whether any particular stochastic process could be used to represent the increments and their observed statistics. This process is one of the basic operations behind the use of non-stationary stochastic processes with stationary increments in real-world hydrologic and hydraulic problems.

Within the context of the present application, for a family of PDFs with zero mean to serve as a basis for representing a set of increment distributions for all possible increments (an infinite number theoretically), the family must also contain an infinite number of members and be closed under additions. This means that if any number of the family members is added together and the area under the resulting curve is re-normalized,

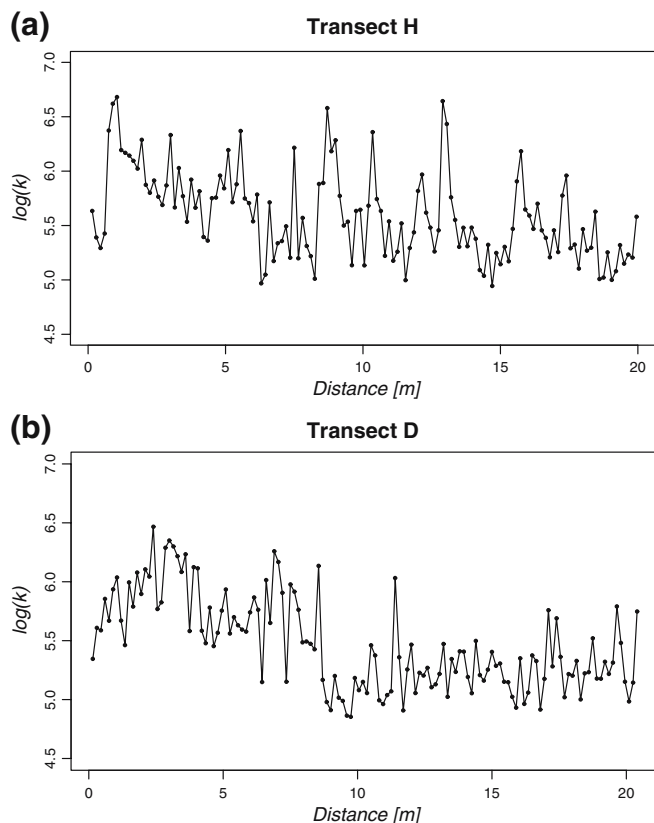


Fig. 2 $\log(k)$ data (k = intrinsic permeability) for transects **a** H and **b** D, both in the same sandstone facies

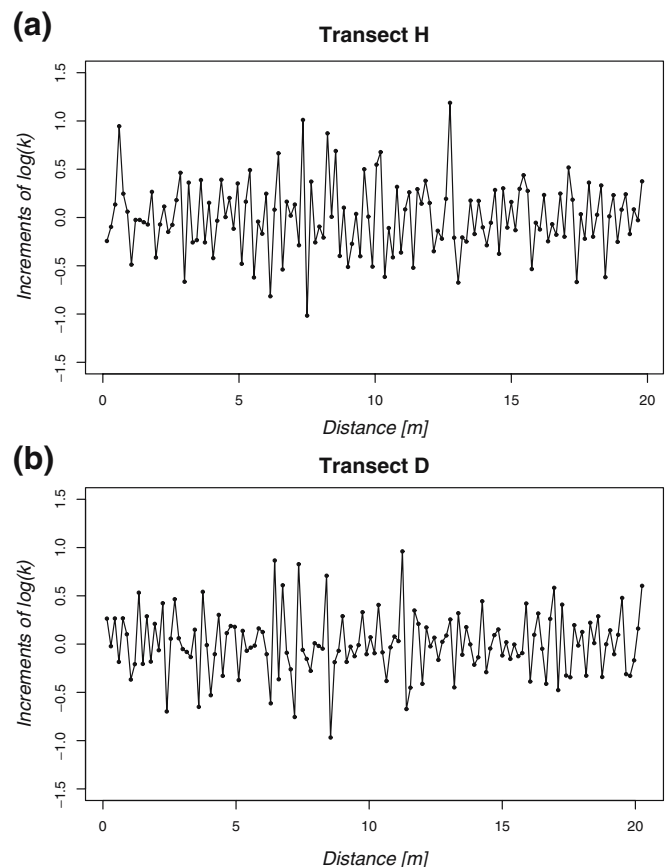


Fig. 3 Plots of $\log(k)$ increments at a 15 cm lag as a function of distance along transects **a** H and **b** D (Castle et al. 2004)

then another member of the family is always produced. It is well known that the Gaussian PDF produces such a family, and the “closed under additions” property is also known conversely as infinite divisibility (any member may be sub-divided into the sum of any number of other members).

The fundamental property of the Gaussian distribution upon which infinite divisibility is based may be stated as follows: the PDF of the sum of n independent Gaussian random variables is still a Gaussian variable with a variance given by

$$\sigma^2 = \sum_{i=1}^n \sigma_i^2, \quad (1)$$

where σ_i is the standard deviation of the i th PDF. In order to make Eq. (1) more applicable, the increments of a property, $\log(k)$, over a lag, h , $\equiv \Delta_h \log(k)$ are considered. Then the variance of the increments for a given h is $< [\Delta_h \log(k)]^2 >$, where the brackets denote the expected value (mean of the squared increments). In order to change from a set of increments of lag h to a set of lag $2rh$, each neighboring pair of increments are added, that is, for each i : $\log(k)_{i+2} - \log(k)_i = (\log(k)_{i+2} - \log(k)_{i+1}) + (\log(k)_{i+1} - \log(k)_i)$. Combining this with Eq. (1) yields the scaling relation:

$$\begin{aligned} < [\Delta_{2h} \log(k)]^2 > = < [\Delta_h \log(k) + \Delta_h \log(k)]^2 > \\ = < [\Delta_h \log(k)]^2 > + < [\Delta_h \log(k)]^2 > = 2 < [\Delta_h \log(k)]^2 >, \end{aligned} \quad (2)$$

where the increment statistical independence makes the cross-terms in Eq. (2) zero. For a change from lag h to an arbitrary lag of rh , Eq. (2) obviously generalizes to

$$< [\Delta_{rh} \log(k)]^2 > = r < [\Delta_h \log(k)]^2 >. \quad (3)$$

As written, the scaling relation implied by Eq. (3) applies to Gaussian distributions with independent (uncorrelated) increments, which are the increments of classical Brownian motion, not fractional Brownian motion. Exhibited by Eq. (3) is “statistical self-similarity of the increments”, because the variance of the increments at lag rh is directly proportional to the variance of the increments at lag h . Mandelbrot and van Ness (1968) extended Eq. (3) to include the case of correlated increments. The result is

$$< [\Delta_{rh} \log(k)]^2 > = r^{2H} < [\Delta_h \log(k)]^2 >, \text{ with } 0 < H < 1. \quad (4)$$

The above generalization is consistent with the existence of long-range positive correlation ($H > 1/2$) or negative correlation ($H < 1/2$) in the property increments, with the classical case of independent increments given by $H = 1/2$. H is called the Hurst coefficient, and when correlation is present, the increments constitute fGn. The non-stationary sums of the increments are fBm. The scaling property exhibited by Eq. (4) is called “statistical

self affinity”, in recognition that the variance of the increments at lag rh is no longer directly proportional to the variance of the increments at lag h . What would be the proportionality constant r now varies as a power function r^{2H} . So if the increment-correlated Gaussian family were to successfully represent the measured increment family at the Utah site (assuming a sufficiently large number of measurements were available), increment PDFs would have to be Gaussian for all selected measurement lags, and the variance of the increments for all lags would have to have the variance scaling relation represented by Eq. (4) with a unique H value. As discussed in a variety of publications (Liu and Molz 1997; Painter 2001; Meerschaert et al. 2004), the $\log(k)$ increment PDFs appear to be non-Gaussian for the smaller lags, and only approach Gaussian at the larger lags. Because of the varying PDF observed in measurements, in addition to the unrealistic infinite variance feature, the Lévy stable-based PDF family also fails as a convincing model for the $\log(k)$ increments, even though that PDF family tends to peak more around the mean than the Gaussian PDF peaks around the mean.

The new generalized Laplace stochastic fractal

Until recently, infinite divisibility of the mean zero Laplace distribution with variance σ^2 and PDF

$$\text{LPDF}(x) = \frac{1}{\sqrt{2}\sigma} e^{-\sqrt{2}|x|/\sigma}, -\infty < x < \infty, \quad (5)$$

was not often used in stochastic modeling. However, because of this property, the Laplace distribution can be used to generate a PDF family that is closed under additions of independent terms. Because the mathematics is more involved than for the Gaussian case, results analogous to those derived above will be mostly described with references given here. In Kotz et al. (2001), it is shown that for the case of uncorrelated lag h increments with mean zero, members of the generalized Laplace family (GLPDF), which is closed under additions, have PDFs of the form

$$\begin{aligned} \text{GLPDF}(x) = & \frac{\sqrt{2}}{\sigma^{h+1/2} \Gamma(h/\nu) \sqrt{\pi}} \left(\frac{|x|}{\sqrt{2}} \right)^{h/\nu-1/2} \\ & K_{h/\nu-1/2} \left(\sqrt{2}|x|/\sigma \right); x \neq 0 \end{aligned} \quad (6)$$

where $h > 0$ is the lag, $\sigma > 0$ is the scale parameter of the PDF, $\nu > 0$ is an additional shape parameter, representing the lag at which the increments have the Laplace PDF Eq. (5), K is the modified Bessel Function of the third kind with index $h/\nu - 1/2$, and Γ is the standard Gamma function. When h is large compared to ν , the distribution of the lag h increments is closer to the normal (Gaussian) distribution, whereas small values of h (relative to ν) bring the increments closer to the Laplace model. Similar to the

Brownian motion, the variance scales linearly with h , being $h\sigma^2/v$ at lag h (so that σ^2 is the variance at lag v).

Although it is not obvious, the probability distribution with PDF given in Eq. (6) is a scale mixture of normal distributions, that is, a lag h increment, considered to be a random variable, can be represented as $\Delta_h \log(k) = G_{h/v}^{1/2} N$, where G_r refers to a gamma random variable with the PDF

$$f(x) = \frac{1}{\Gamma(r)} x^{r-1} e^{-x}, x > 0, \quad (7)$$

and N is an independent normal variable with mean zero and standard deviation σ . Recall that in the case of Brownian motion, a similar representation results where instead of the stochastic term $G_{h/v}$, a non-stochastic scale h results. This stochastic representation of the uncorrelated fLam increments leads to many important properties of this process, including moments, tails, and scaling behavior (see, e.g., Kotz et al. 2001). In particular, the scaling relations for Eqs. (1), (2), and (3) apply for the uncorrelated fLam as well.

In case of a correlated fLam (see Meerschaert et al. 2004; Kozubowski et al. 2006), the lag h increments are given by $\Delta_h \log(k) = G_{h/v}^H N$, where $G_{h/v}$ and N are as above and H is the Hurst coefficient. The uncorrelated case corresponds to $H=1/2$. Similarly to fBm, fLam exhibits long-term positive correlation when $H>1/2$, and (short-term) negative correlation when $H<1/2$. Both fLam and fBm increments have moments of all orders, unlike the Lévy-stable fractals, where the variance is always infinite. One difference between fLam and fBm is their scaling behavior. For fLam, the relation between the variance of the increments at lag rh to that of the increments at lag h is of the form (Kozubowski et al. 2006)

$$\langle [\Delta_{rh} \log(k)]^2 \rangle = \frac{\Gamma(2H + rh/v) \Gamma(h/v)}{\Gamma(2H + h/v) \Gamma(rh/v)} \langle [\Delta_h \log(k)]^2 \rangle, \quad (8)$$

with $0 < H < 1$.

In the uncorrelated case, this equation reduces to Eq. (3). Although a power law scaling is not exactly present, this relation will coincide with Eq. (4) as the lag h increases. It is also worth noting that whereas all increments of fBm have Gaussian PDFs, only lag $h=v$ increments of fLam (in the uncorrelated case) are actually Laplace PDFs given by Eq. (5)—other increments being generalized Laplace PDFs as in Eq. (6). In the correlated cases, none of the increment PDFs are precisely Laplace. However, in all cases for $h/v < 1$, the increments of fLam are highly peaked around the mean with fat tails, whereas for $h/v > 1$, the peaking gradually disappears and the increments converge to a Gaussian PDF as h increases (because the gamma distributed term $G_{h/v}^H$ in the above stochastic representation behaves like h/v as h approaches

infinity). As discussed in Meerschaert et al. (2004), this is the type of behavior that is being observed qualitatively in $\ln(k)$ data sets, as well as velocity increment data sets obtained from turbulent flow experiments. (It is now known, however, that the present model does not fit turbulent velocity increment data quantitatively in the non-Gaussian range.) Shown in Fig. 4 are members of the generalized Laplace family (PDFs of fLan: uncorrelated case $H=1/2$).

Gaussian and generalized Laplace stochastic fractals fitted to the Utah data

The Utah data presented in Fig. 3a,b (with empirical distribution shown in Fig. 1) can be modeled as either Brownian motion (fBm with Hurst coefficient $H=1/2$) or as Laplace motion (fLam with $H=1/2$). According to both models, the lag h increment data correspond to *independent* (uncorrelated) observations from either Gaussian or generalized Laplace distributions. The data from the two transects, taken at the lag of 0.15 m, were combined resulting in 267 total measurements.

According to the Gaussian model, the lag $h=0.15$ -m increments are a random sample from a normal distribution with variance $0.15\sigma^2$. Consequently, the parameter σ can be estimated as a sample standard deviation divided by 0.15. When applied to these data, this results in $0.35/0.15=2.33$. In Fig. 5b, a sample generated from this model is presented. As expected, not as many relatively small and large values are seen in comparison to the empirical data (see Fig. 3a,b). This result is expected because of the relatively large value (3.66) for the kurtosis of the Utah data.

The uncorrelated Laplace model assumes the increments have a generalized Laplace PDF Eq. (6). The

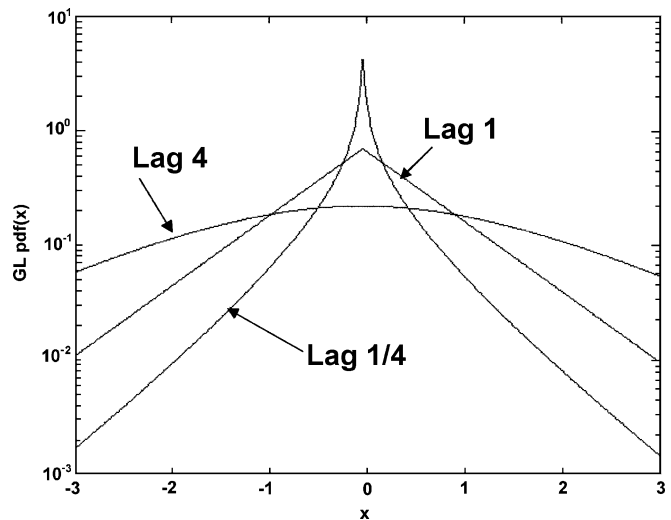


Fig. 4 Semi-log plots of selected members of the infinitely divisible Laplace family for the case of uncorrelated increments. The two straight lines represent the Laplace (double exponential) distribution, which corresponds to a lag of 1. Notice how the shapes of the various members change dramatically with lag

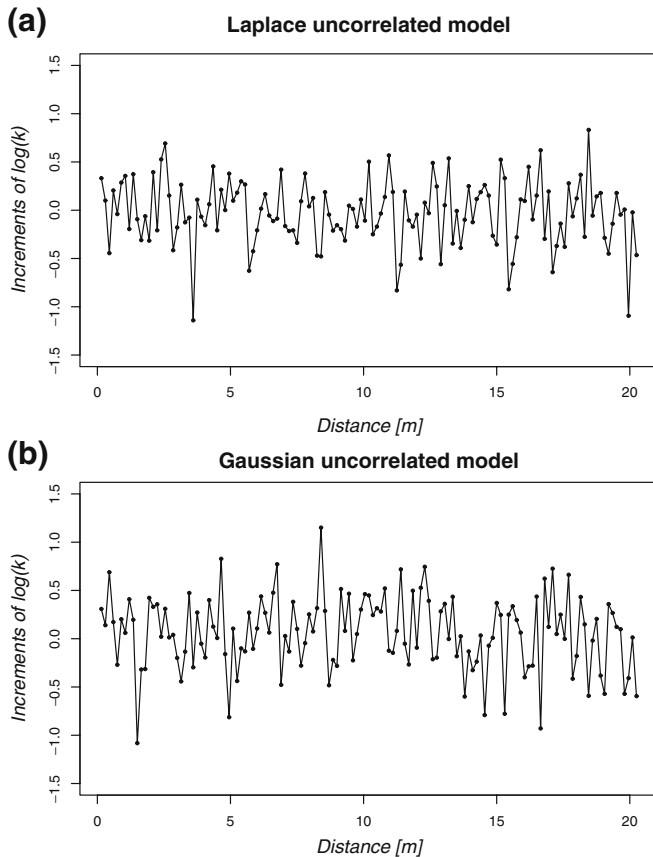


Fig. 5 Sample realizations of an uncorrelated **a** fLan and **b** fGn with the parameters estimated from Utah data presented in Fig. 3a,b

parameters can be estimated with the method of moments utilizing formulas for variance and kurtosis corresponding to Eq. (5), see, e.g., Kotz et al. (2001). These parameters are

$$s^2 = \sigma^2 h / \nu \text{ and } \kappa = 3 \left(1 + \frac{\nu}{h} \right),$$

for the Laplace model and the methods of moments, respectively. Using the empirical values of $\hat{s}=0.35$ and $\kappa=3.66$ obtained from these data, along with $h=0.15$, in the above expressions, results in estimates of $\sigma=0.157$ and $\nu=0.03$. Figure 6 compares the model results with the data. It is clear that from a distributional point of view, fLan agrees with the data far better than fGn, both at the tails and at the peak above zero. However, the data are not symmetrical. A random sample from such a model is presented in Fig. 5a. Unlike uncorrelated fGn, fLan more consistently represents the data (presented in Fig. 3a,b) as it fluctuates more widely around the mean.

Although the generalized Laplace model Eq. (6) provides a good fit to the marginal empirical distribution of the Utah data, the model does not account for possible correlations of the data. A plot of the sample autocorrelation function of the data (Fig. 7) shows that not all correlations are negligible. The fBm does take correlations into account, but it comes with one-dimensional Gaussian

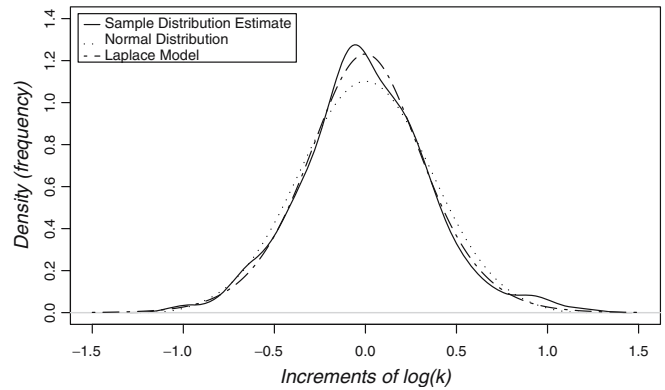


Fig. 6 Empirical distribution of the lag 15 cm $\log(k)$ increment data (shown in Fig. 1) along with a fitted generalized Laplace and Gaussian PDFs

distributions, which were shown not to be appropriate for these data due to the high kurtosis.

Another correlated model is the fLan discussed above. By an appropriate choice of the Hurst parameter $H < 1/2$, the negative correlation at lag 0.15 shown in Fig. 7 can be obtained. With this model, the values of the kurtosis, variance, and lag h correlation of the increments of $\log(k)$ are, respectively,

$$\begin{aligned} \kappa &= 3 \frac{\beta(2H, h/\nu)}{\beta(2H, 2H+h/\nu)}, \\ s^2 &= \sigma^2 \frac{\Gamma(2H+h/\nu)}{\Gamma(h/\nu)}, \\ \rho(h) &= \frac{\beta(2H, 2h/\nu)}{2\beta(2H, h/\nu)} - 1 \end{aligned}$$

where $\beta(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the standard beta function, see Kotz et al. (2001) and Kozubowski et al. (2006). The first and the third equations for H and ν are solved using the estimated values $\kappa=3.66$ and $\rho(h)=-0.339$, followed by estimating σ from the second equation using the empirical value of $\hat{s}=0.35$. The numerical solution of the first and third equations yields $H=0.114$ and $\nu=0.4$

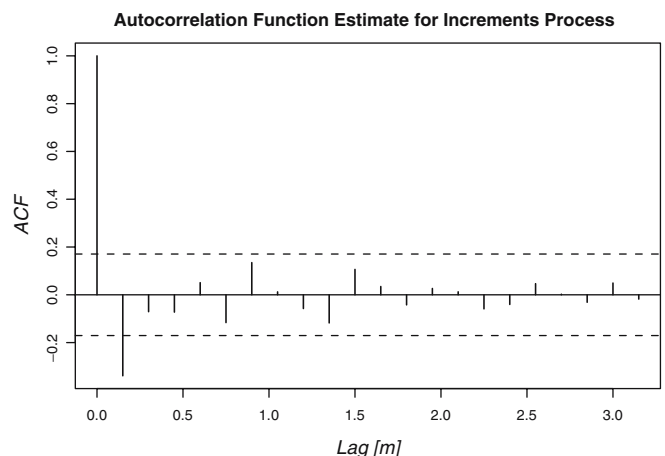


Fig. 7 Sample autocorrelation functions of Utah data (presented in Fig. 3a,b) showing statistically significant negative correlation at lag 0.15 (outside lower horizontal dashed line)

(details available upon request), leading to $\sigma=0.44$. Figure 8 shows that the empirical autocorrelation function and the theoretical function (corresponding to the fLan with the above parameters) are similar.

In Fig. 9, which is analogous to Fig. 6 corresponding to the uncorrelated models, the empirical density of the Utah data are compared with the fitted densities of correlated fLan and fGn. Once again, the model corresponding to the Laplace distribution can capture the peak and the tails of the empirical data much better than the model based on the Gaussian distribution. Although more data are desirable, the Utah field data can be modeled by fLan in a reasonable manner. Ultimately, the PDFs would become Gaussian at sufficiently large lags. Note that the parameter estimates of the correlated fLan are different from those of the uncorrelated estimates (ν : 0.15 vs. 0.4 and σ : 0.157 vs. 0.44). This result shows the importance of proper modeling of the dependence structure.

Finally, Fig. 10a,b illustrates random realizations of correlated fLan and fLan with the parameters estimated above. The method of simulation utilizes a representation of fLan as an fBm subordinated to a gamma process, as discussed in Kozubowski et al. (2006). This method consists of first obtaining a sample from a gamma process (generators of gamma random variates are readily available in standard software packages), followed by an application of the innovation algorithm to generate values of fBm evaluated at gamma process points. As can be seen, the simulations appear to be similar to the actual empirical data presented in Figs. 2a,b and 3a,b. Many of the $\log(k)$ data sets studied over the past decade display similar features when measurements are made on core plugs or with the gas mini-permeameter (Painter 1996, 2001; Molz et al. 2004; Meerschaert et al. 2004). Larger data sets are needed, however, in order to make a more definitive conclusion.

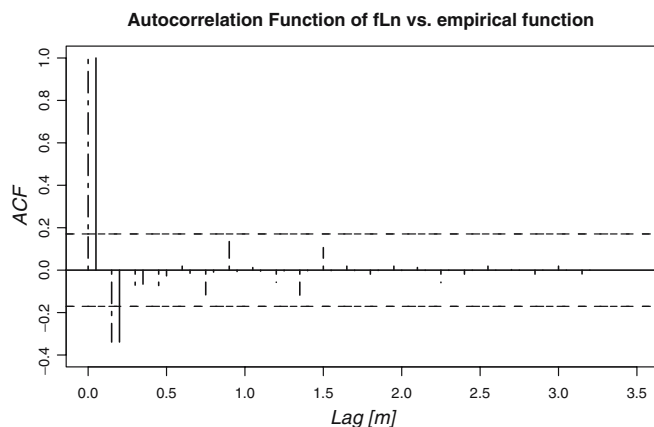


Fig. 8 The autocorrelation function of the fitted fLan (vertical solid lines) compared with the empirical function obtained from the Utah data (vertical broken lines). The autocorrelation values falling within the confidence limits (horizontal dashed lines) are not significantly different from zero

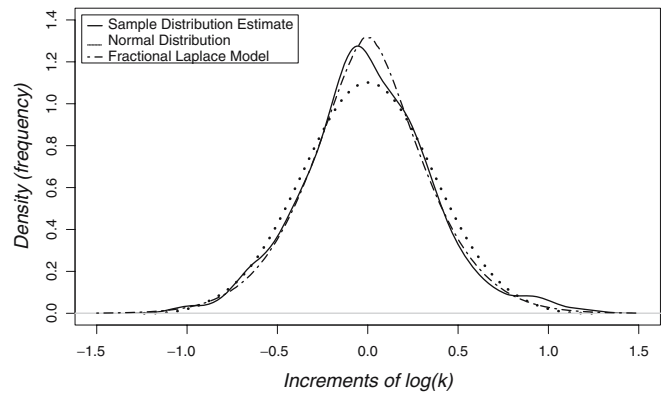


Fig. 9 Empirical distribution of the lag 15 cm $\log(k)$ increment data (shown in Fig. 1), along with fitted densities of correlated fLan and fGn

Discussion and conclusions

In Lu et al. (2002), it was argued that in developing stochastic fractal models of subsurface heterogeneity, data from different facies should not be mixed. Potential problems were illustrated using data from four facies of an alluvial fan (floodplain, levee, channel deposits, and debris flow deposits). Assumed Gaussian $\log(K)$ incre-

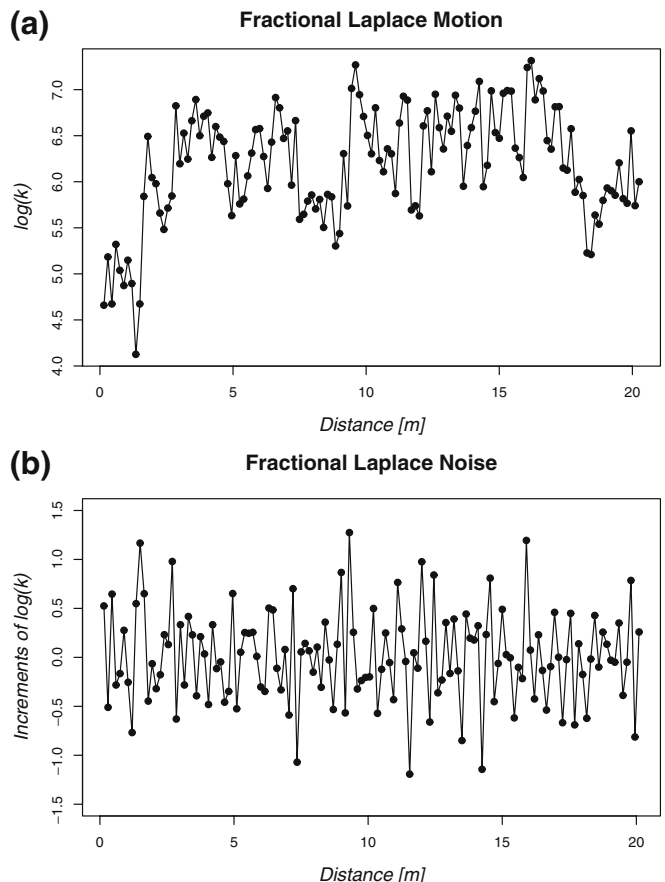


Fig. 10 Computer generated **a** fLan and **b** fLan with the Hurst coefficient $H=0.114$, $\sigma=0.44$, and $\nu=0.403$ (short-term dependence, negative correlation)

ment variation within each facies could lead to the conclusion that, overall, the alluvial fan data were displaying fractional Lévy motion behavior. In the Lu et al. illustration, Gaussian (fBm) $\log(K)$ increment behavior was emphasized within each facies, even though there was evidence that non-Gaussian behavior is observed as lag size diminishes. A model for $\log(K)$ increment PDFs that produces non-Gaussian behavior at smaller measurement lags and converges to Gaussian behavior at larger measurement lags, all within a single facies, is presented here. The model is based on the classical Laplace PDF and its generalizations. The interesting properties of these generalizations are not widely known, but presently they are being more widely disseminated. The new stochastic fractal, non-stationary fLam with stationary increments fLan, is presented by contrasting its properties with those of the well-known Gaussian stochastic fractal, non-stationary fBm with stationary increments fGn, and then presenting its properties, both mathematically and with illustrations. Mathematical details are contained in key references (Kotz et al. 2001; Meerschaert et al. 2004, and Kozubowski et al. 2006). As illustrated in Fig. 4, this fractal is different because the character of the underlying increment PDFs changes dramatically with lag size. These changes lead to lack of self-similarity and self-affinity as they are traditionally defined, and data also appear to display this characteristic. In the larger lag size ranges, however, approximate self-affinity does hold, and fLan converges to fGn. This type of scaling is also embodied explicitly in Eq. (8). Convergence to fGn versus lag size (h) is controlled by the rescaling parameter ν .

The fractals fLam/fLan are considered to be a class of stochastic fractals because variations occur on all scales (h values) and statistical parameters change in an orderly manner with h . In addition, these fractals arise from the theory of non-stationary stochastic processes with stationary increments, the basis for virtually all stochastic fractals. In most presentations involving fractals, the so-called fractal dimension is important. In stochastic fractals, however, the concept of auto-correlation, as defined by the Hurst coefficient H , can be used in place of fractal dimension. Because fractal-like behavior is apparently always observed over a finite range of scales (Avnir et al. 1998), and a “true” fractal having a “true” fractal dimension would seem to require an infinite range of scales, auto-correlation of increments may be the preferable concept when dealing with applications of stochastic fractals of the type discussed here over a limited range of lags.

The PDFs for the uncorrelated, generalized, Laplace family are given by Eq. (6). For the correlated case, which is of main practical interest because $\log(k)$ increments are typically correlated, a closed form expression for the PDF is not known. In Kozubowski et al. (2006), however, an integral form is given, and this form may be used to numerically calculate the PDFs of interest. Thus, the main field procedure for further necessary testing of the

fractional Laplace theory, and any other of the existing fractal theories, is to measure increment distributions along transects, calculate frequency distributions from the data, and compare results to various members of the auto-correlated fLan family, or any other family of interest. The variances of the frequency distributions should also change with lag size (scale) according to Eq. (8). There are mathematical reasons such as the geometric central limit theorem (Kotz et al. 2001), for surmising that fLam/fLan may be more fundamental than other approaches that have been proposed for modeling $\log(k)$ frequency distributions such as the flexible scaling model of Painter (2001). If this result turns out not to be the case, then other approaches may be preferable.

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