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Estimation of magnetotelluric transfer functions in the time domain over a wide frequency band

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SUMMARY

Determination of the transfer function is the main objective in magnetotelluric (MT) and geomagnetic data processing. In this paper we describe a new algorithm for finding a transfer function through determination of the impulse response in the time domain. The proposed method was developed for the long time-series with which we are usually dealing in short-period recording. It allows the determination of the transfer function for a broad range of frequencies, the only limitation being the duration of recording. In contrast to other methods for determination of the time-domain transfer function we selected an approximation of impulse response in which the number of unknowns to be determined is proportional to the frequency range of interest on a logarithmic scale. This is a natural way to deal with functions such as the MT impedance tensor or geomagnetic vector. Our approximation makes it easy to find the predicted electric field, which is very useful for assessing the quality of the recorded data.

Key words: electromagnetic methods, geomagnetism, magnetotellurics, numerical techniques.

1 INTRODUCTION

Electromagnetic induction experiments are used to determine the electrical resistivity structure of the Earth. If the vertical component of the external magnetic field producing the induction effect is zero (the incident wave is a plane wave), then in the Cartesian coordinate system there are the following linear relations in the time domain between the electromagnetic field components for a fixed point at the Earth's surface:

$$\begin{aligned}\tilde{z}(t) &= \int_{-\infty}^{\infty} \tilde{a}(\tau)\tilde{x}(t-\tau)d\tau + \int_{-\infty}^{\infty} \tilde{b}(\tau)\tilde{y}(t-\tau)\,d\tau \\ &= \tilde{a}(t) * \tilde{x}(t) + \tilde{b}(t) * \tilde{y}(t),\end{aligned} \tag{1}$$

where $\tilde{x}(t)$, $\tilde{y}(t)$ are the horizontal magnetic field components H_x, H_y and $\tilde{z}(t)$ is the vertical magnetic component H_z in the case of geomagnetic sounding or one of the horizontal electric components E_x, E_y for magnetotelluric sounding. In the frequency domain, after using the Fourier transform, we get, respectively,

$$\tilde{Z}(\omega) = \tilde{A}(\omega)\tilde{X}(\omega) + \tilde{B}(\omega)\tilde{Y}(\omega)\,.$$

In the above equation the capital letters stand for the complex spectra, the Fourier transforms of the respective functions of eq. (1). Functions $\tilde{a}(t)$ and $\tilde{b}(t)$ in the time domain are called impulse responses, and their transforms in the frequency domain are the transfer functions. Weidelt & Kaikkonen (1994) proved that for 2-D structures all the *B*-polarization impedances (TM mode) are of minimum phase and impulse responses are causal. For complex 3-D structures causality does not necessarily hold in general. Experience shows, however, that for most real data the causality is fulfilled. Methods for determining the time-domain transfer function allow (in contrast to frequency-domain methods) us to take this condition into account in a simple way. We assume that the impulse responses from eq. (1) satisfy the causality condition, i.e. they are zero for $t < 0$. We also consider a case when the above assumption fails and non-causal impulse responses are needed.

The first step of electromagnetic investigation is to use eqns (1) or (2) to determine the impedance tensor and geomagnetic vector from the electromagnetic field components recorded at the Earth's surface. Most theoretical studies and computational methods derived from them are limited to the frequency domain, with total negligence of the time domain. This stems from the fact that spectral methods are widely used in engineering (not only in geophysics) and there is an ample literature on this subject. There are, however, several problems relating to the processing of frequency-domain data resulting from the fact that the time-domain recordings are the starting point. The need to adopt numerous subjective assumptions, such as the selection of the window, section length or the data whitening method, is the reason why the results obtained are not always satisfying. Although spectral methods have been improved so that they are able to better cope with various types of noise, determination of the time-domain transfer function may provide a useful alternative or complement which may offer significant

advantages when noise is localized in time. Probably a hybrid method, a combination of time- and frequency-domain data processing, should be the best way to estimate the transfer function. Larsen's approach to estimation of the frequency-domain transfer function (Larsen *et al.* 1996) is an example of method in which outlier detection is also done in the time domain.

There are a few papers dealing with the determination of impulse response in the time domain from eq. (1). Following the paper by Kunetz (1972), time-domain methods have been developed by Wieląd ek & Ernst (1977), McMechan & Barrodale (1985), Yee *et al.* (1988), Spagnolini (1994), Santarato & Spagnolini (1995) and Wieląd ek & Nowożyński (2001). In these papers the impulse responses have been approximated by low-order trigonometric polynomials or their quotients. Such approximations are sufficiently accurate when the observation sequences contain hundreds or thousands of samples. In the case of short-period MT recordings with millisecond sampling intervals, when we have at our disposal a large number of data (millions of samples) and the maximum periods recorded are comparable to the duration of recording, the use of such methods may be problematic. The reason is that the degree of the approximation polynomials used determines the range of frequencies which can be present in the data. Increasing the degree of the polynomials may cause the problem to become nearly singular with respect to a large number of the unknowns to be determined. If we use a rational approximation in the form of a polynomial quotient it may cause the polynomial zeros in the denominator to be placed too close to the unit circle on the complex plane. To avoid such a situation, we must divide the data into several frequency ranges by filtration and changing the sampling interval and carry out independent calculations in every range. This may give rise to misfit of the results on the boundaries of the bands. In the present paper we describe a time-domain method for determination of the impulse response which is free of the previously mentioned drawbacks. We choose an impulse response approximation which is defined so that for each frequency decade there are five to ten unknowns to be determined.

Reliable estimation of the transfer function depends not only on the method of approximation but also on the quality of the data. Two options can be used to improve data quality. We can either carefully select the data, i.e. examine the recordings and choose the fragments that we think are not very disturbed, or make the process automatic. An example of the latter is the time-domain directional noise cancelling (DNC) technique proposed by Santarato & Spagnolini (1995). The noise reduction techniques use some characteristic of the noise that makes it distinct from the signal and is to a lesser degree related to the choice of impulse response approximation. The transfer function calculation method described in this paper uses a very simple iteration technique, making it possible to eliminate short-duration noise, notably spikes. One may also consider the use of more complex techniques, such as DNC.

2 DESCRIPTION OF THE PROBLEM FOR DISCRETE VALUES

A function $f(t)$ is said to be band limited if its complex spectrum $F(\omega) = 0$ for $|\omega| \geq \Omega$, where $\Omega > 0$, is fixed. Band-limited functions play an important role in data acquisition, because the sampling theorem states that band-limited function $f(t)$ can be uniquely reconstructed from its discrete values $f_i = f(t_0 + i\,\Delta t)$ (for any fixed t_0), provided that the sampling interval $\Delta t \leq \pi/\Omega$. Let φ be a band-limited function whose complex spectrum $\Phi = 0$ for $|\omega| \geq \Omega_0$ and amplitude spectrum $|\Phi(\omega)| > 0$ for $|\omega| < \Omega_0$. Functions $x(t) = \varphi(t) * \tilde{x}(t)$, $y(t) = \varphi(t) * \tilde{y}(t)$ and $z(t) = \varphi(t) * \tilde{z}(t)$ are also band limited. We define function $a(t)$ so that its complex spectrum is equal to $\tilde{a}(t)$ for $|\omega| \leq \Omega_0$ and zero for $|\omega| \geq \Omega$ (Ω is fixed and $\Omega > \Omega_0$). The complex spectrum of $a(t)$ may be arbitrary in the interval (Ω_0, Ω). In a similar manner we define function $b(t)$. Then it follows from eq. (1) that the following equation also holds:

$$z(t) = a(t) * x(t) + b(t) * y(t). \tag{3}$$

Making use of the convolution properties and applying the sampling theorem for the sampling interval $\Delta t = \pi/\Omega$ to eq. (3), we obtain the equivalent formula for discrete values

$$z_k = \Delta t \sum_{i=-\infty}^{\infty} a_i x_{k-i} + \Delta t \sum_{i=-\infty}^{\infty} b_i y_{k-i}, \tag{4}$$

where

$x_i = x(t_0 + i\,\Delta t),\ y_i = y(t_0 + i\,\Delta t),\ z_i = z(t_0 + i\,\Delta t),$

$a_i = a(i\,\Delta t),\ b_i = b(i\,\Delta t).$

Expressing $A(\omega)$ and $B(\omega)$ as Fourier series in the interval $[-\Omega, \Omega]$ we have

$$A(\omega) = \begin{cases} \Delta t \sum_{j=-\infty}^{\infty} a_j \mathrm{e}^{-ij\omega\Delta t} & \text{for } |\omega| \leq \Omega \\ 0 & \text{for } |\omega| > 0 \end{cases} \qquad B(\omega) = \begin{cases} \Delta t \sum_{j=-\infty}^{\infty} b_j \mathrm{e}^{-ij\omega\Delta t} & \text{for } |\omega| \leq \Omega \\ 0 & \text{for } |\omega| > 0. \end{cases} \tag{5}$$

Eq. (4) will be used to determine impulse responses $a(t)$ and $b(t)$ and then, from the Fourier series (5), transfer functions $A(\omega)$ and $B(\omega)$. From the definition of functions $x(t)$, $y(t)$ and $z(t)$ it follows that eq. (4) is satisfied for a sampling interval π/Ω_0 provided that $A(\omega)$ and $B(\omega)$ are zero for $|\omega| \geq \Omega_0$. The purpose of choosing a shorter sampling interval (oversampling) is to improve the approximation of the transfer function by a finite Fourier series. At the Nyquist frequency, the finite Fourier series, corresponding to the finite impulse response necessary for a time-domain approach, is continuous and takes a real value, while the transfer function may be complex. Therefore, at a certain point near the Nyquist frequency the approximation of the imaginary part of the transfer function will be contaminated by a large error. Choosing a shorter sampling interval π/Ω we limit ourselves (eq. 4) to the approximation of transfer function in the interval $[-\Omega_0, \Omega_0]$, neglecting the frequencies $|\omega| > \Omega_0$ for which the approximation error may be large.

If we assume that the linear system (1) is causal then, upon passing to the band-limited function, system (4) will cease being causal because any function cannot be simultaneously time and band limited. We can only consider a discrete equivalent of the causality condition assuming that $a_j = a(j\Delta t) = 0$ and $b_j = b(j\Delta t) = 0$ for $j < 0$. This does not mean, however, that $a(t) = 0$ and $b(t) = 0$ for $t < 0$. We may wonder if the discrete impulse response with negative coefficients equal to zero may well approximate a causal system. Egbert (1992) examined this problem: taking as an example a 1-D model limited to a half-space he demonstrated that the discrete values of the impedance transform in the time domain are non-zero for negative indices. However, he did not account for the situation when the sampling interval Δt is lower than π/Ω_0 required in the sampling theorem. If we assume that $\Delta t = \pi/\Omega$, we have at our disposal an additional interval (Ω_0, Ω) in which we can arbitrarily choose the transfer function, e.g. so as to have it at least continuous. In Egbert's example the impedance passes to zero at the Nyquist frequency $\pi/\Delta t$ in a jump-like manner, and this may be the reason why the values for negative coefficients are comparable with those for positive coefficients.

It follows from our attempts at impedance approximations for 1-D models that (provided that the adopted sampling interval is at least a few times less than that defined by the Nyquist frequency) we obtain, under the assumption that the negative coefficients are zero, the approximations whose errors for geophysical applications may be considered small. Nevertheless, negative impulse response coefficients are needed to achieve a good transfer function approximation of a causal system for frequencies close to the Nyquist frequency and, as postulated by Egbert, they should be determined. There is a large disproportion between the number of negative and positive coefficients, though. In Fig. 1 we show a transfer function approximation for a 1-D model (which, as we know, is causal), in which three negative coefficients and a few thousand positive ones were used. A good approximation was obtained for periods longer than three sampling intervals. In the remainder of this paper, we will consider a discrete linear system (4) corresponding to the causal system (1) in which, along with positive coefficients, we will take into account a small number of negative ones. In our method for the determination of the impulse response we will implement the above limitation with regard to the negative coefficients. However, the method itself can in a simple way be generalized to include non-causal linear systems without introducing limitations on the negative coefficients.

The sampling theorem is used in measuring instruments which, instead of the function $g(t)$ in which we are interested, record discrete values of the function $f(t) = \varphi(t) * g(t)$. The frequency response $\Phi(\omega)$ of the instrument takes into account, among other things, filters (analogue or digital) used for zeroing the frequencies in excess of Ω_0 and the properties of the sensors applied.

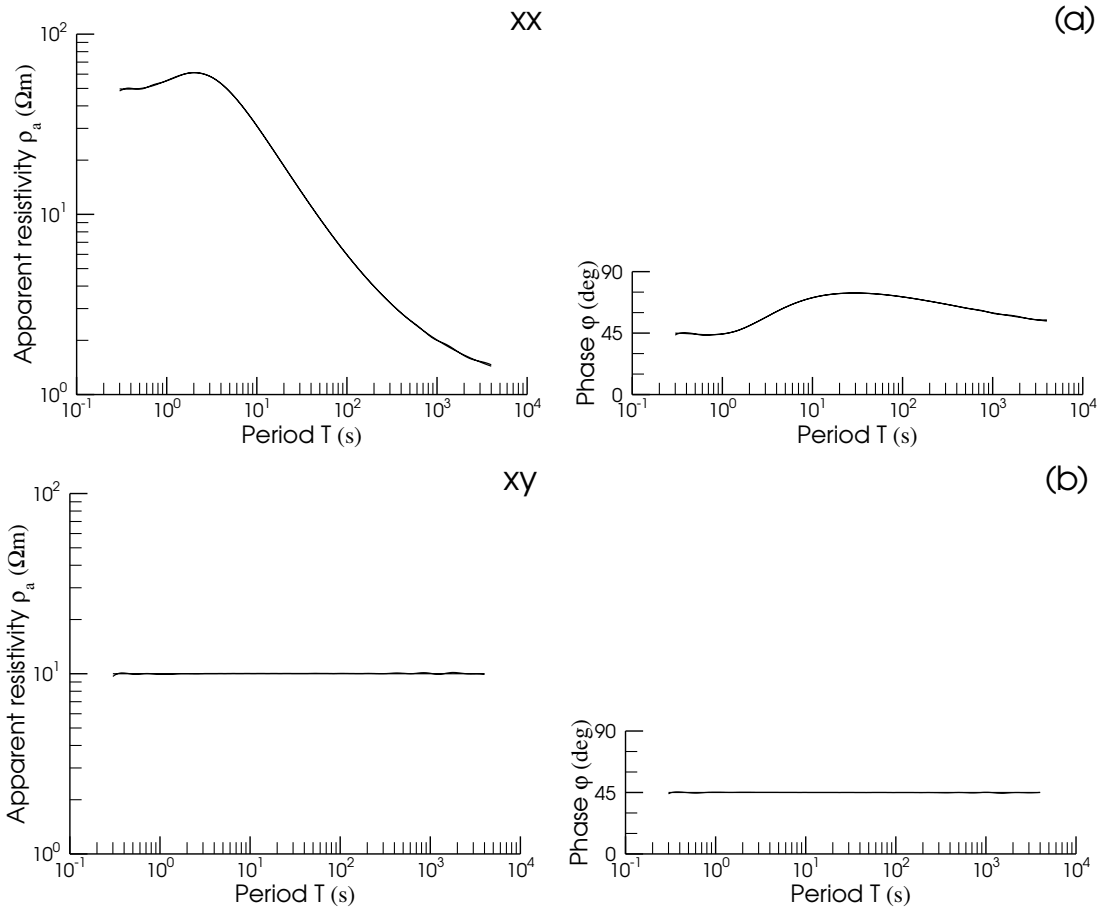

Figure 1. Apparent resistivity ρ_a and impedance phase φ calculated from synthetic data for two one-dimensional models: (a) one-layer model with 50 Ω m, 6 km, 1 Ω m, and (b) half-space with 10 Ω m. The solid line represents the adopted model, the dashed line the calculated one.

If the electromagnetic field is recorded by an instrument that has identical frequency responses for all the components, eq. (4) will also be satisfied for the measured quantities $x(t)$, $y(t)$ and $z(t)$.

3 APPROXIMATE SOLUTION

As a result of electromagnetic field measurements, we obtain finite sequences $\{x_i\}_{i=0}^{n-1}$, $\{y_i\}_{i=0}^{n-1}$, $\{z_i\}_{i=0}^{n-1}$ (in a certain time interval defined by the duration of recording) of discrete values of functions $x(t)$, $y(t)$ and $z(t)$ satisfying eq. (4), where Δt is the sampling interval of the measuring instrument, t_0 is the start time of recording and $n\Delta t$ is the duration of the recording. We assume that the impulse response approximation will be made with $m = m_1 + m_2$ non-zero coefficients, where m_1 and m_2 denote the numbers of negative and non-negative coefficients respectively. We also assume that the values of the recorded fields are known with an accuracy up to a constant s, which must be an unknown quantity. Under such assumptions, eq. (4) takes the form

$$z_k + \varepsilon_k = \Delta t \sum_{i=-m_1}^{m_2-1} a_i x_{k-i} + \Delta t \sum_{i=-m_1}^{m_2-1} b_i y_{k-i} + s \quad \text{where} \quad k = m_2 - 1, \ldots, n - m_1 - 1. \tag{6}$$

In the above equation, ε_k is the sum of approximation errors, instrumental errors, industrial noise and departure from the assumption that the vertical component of the magnetic field evoking the electromagnetic induction is zero (the incident wave is a plane wave). Linear equations such as eq. (6) usually also take into account a term related to linear or even higher-order trends that may appear in the course of recording. Moreover, taking account of the trend may reduce the error ε_k if the approximation applied is not sufficient for periods greater than the duration of recorded interval. However, we think that much better results will be obtained if instead of determining the trends we use long-period filtration of data, since by taking into account trends we are able to accomplish only an approximate elimination of the long periods from eq. (6). When the data are cleaned of trends, the mean values of sequences $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$ are zero and the constant s in eq. (6) could be neglected. However, such a procedure is not formally equivalent to the determination of s from the condition of a minimum of the error ε. Since the presence of s does not substantially increase the cost of the method (one additional unknown), we will retain this quantity. The method we describe can also be used for data with errors of short-lived large amplitudes, which will automatically be eliminated through rejection of the equations corresponding to them. Depending on which fragments of the recording are to be neglected, the mean value may vary, so it is better to retain s as an unknown. The impulse response length m should be selected so as to be comparable with the longest periods occurring in the data and not to exceed, say, $n/3$, since in order to determine the impulse response it remains to solve $n - m + 1$ equations. To simplify notation (and for this reason alone) we will take into account one time-series of data.

4 ALGORITHM FOR DETERMINATION OF THE IMPULSE RESPONSE

For convenience, in the following we will refer to a band-limited function through its dependence on time or through a sequence of its discrete values. Moreover, whenever discussing relations for a pair of functions $a(t)$ and $x(t)$, we will tacitly assume that similar relations hold for $b(t)$ and $y(t)$.

An attempt at determining all coefficients $\{a_i\}$ and $\{b_i\}$ from eq. (6) is useless because there are too many of them. To reduce the number of unknowns, we will express impulse responses $a(t)$ and $b(t)$ as linear combinations of $l - l_p$ band-limited base functions $\psi_j(t)$ and functions $\chi_j(t) = h(t - j\Delta t)$:

$$a(t) = \sum_{j=l_p}^{l-1} \alpha_j \psi_j(t) + \sum_{j=-m_1}^{m_3-1} \gamma_j \chi_j(t), \quad b(t) = \sum_{j=l_p}^{l-1} \beta_j \psi_j(t) + \sum_{j=-m_1}^{m_3-1} \delta_j \chi_j(t), \tag{7}$$

where

$$h(t) = \frac{1}{\pi t} \sin\left(\frac{\pi}{\Delta t} t\right) = \frac{1}{\Delta t} \operatorname{sinc}\left(\frac{\pi}{\Delta t} t\right).$$

For non-causal systems we should take into account unknowns α_j and β_j with negative indices and the base functions $\psi_j(t) = \psi_{-j}(-t)$ corresponding to them. Unknowns γ_j and δ_j were introduced in order to improve the approximation close to the Nyquist frequency. Index l_p is the smallest number for which the base functions $\psi_j(t)$ for $j \geq l_p$ and functions $\chi_j(t)$ for $j \geq 0$ form a linearly independent system. Upon passing to discrete values and substituting the so defined $a(t)$ and $b(t)$ into eq. (6) we obtain a linear problem from which, using for example the least-squares method, we can determine the unknowns α_j, β_j, γ_j, δ_j and s. The success of the method relies upon the correct choice of function $\psi_j(t)$: for a small l, it should ensure a good approximation of impulse responses and make it possible to calculate at a small cost the convolutions $\psi_j(t) * x(t)$ and $\psi_j(t) * y(t)$ needed for forming the columns of the least-squares problem matrix.

Let $\{k_j\}$ be an increasing sequence of natural numbers ($k_0 = 0$, $k_1 = 1$) and the band-limited functions $h_j(t)$ be defined as follows:

$$\begin{aligned} h_0(t) &= h(t), \\ h_j(t) &= \frac{1}{4}h(t) + \frac{1}{2}h(t - k_j\Delta t) + \frac{1}{4}h(t - 2k_j\Delta t). \end{aligned} \tag{8}$$

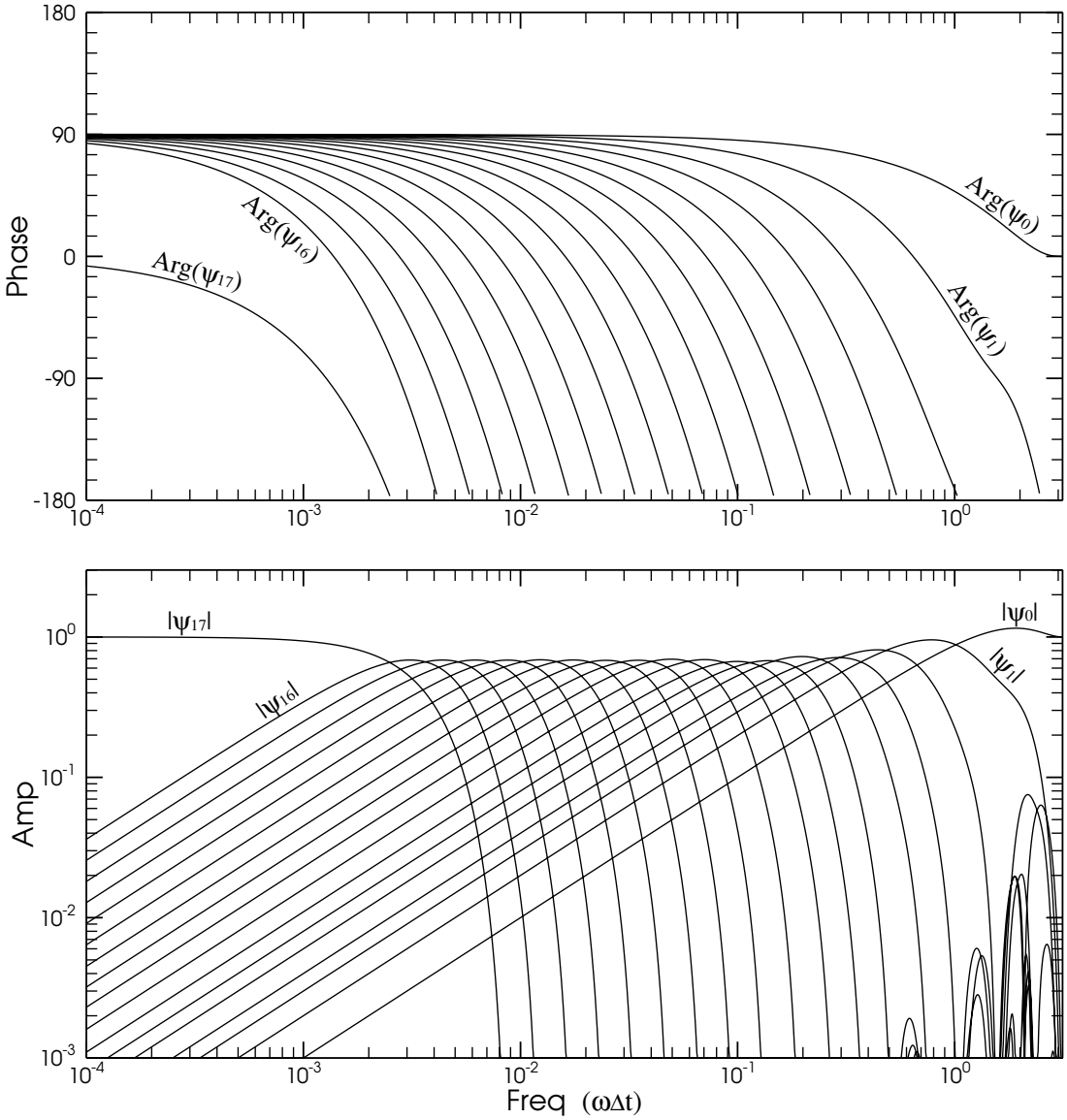

Figure 2. Amplitudes and phases of the Fourier transform of functions $\{\psi_j\}$ whose linear combinations form the transfer function, for $q = \sqrt{2}$.

The base functions $\psi_j(t)$ will be defined by the relations

$$\begin{aligned} \psi_j(t) &= h_0(t) * \cdots * h_j(t) - h_0(t) * \cdots * h_{j+1}(t), \quad j = 0, \ldots, l-2; \\ \psi_{l-1}(t) &= h_0(t) * \cdots * h_{l-1}(t). \end{aligned} \tag{9}$$

The sequence $\{k_j\}$ is formed from the geometrical sequence $\{q^j\}$ $(q > 1, j > 0)$

$$k_j = \left\lfloor q^{i_j - 1} + \frac{1}{2} \right\rfloor, \tag{10}$$

after removing the repeating values, which can occur when $q < 2$.

The above definition of base function $\psi_j(t)$ can be explained as follows. Function $h_1(t)$ is a simple low-pass filter. Convolutions $h_0(t) * \cdots * h_j(t)$ are low-pass filters whose bandwidth decreases in a geometrical progression, as a result of the definition of sequence $\{k_j\}$. Hence, base functions $\{\psi_j(t)\}$, being the differences of such convolutions, act as filters possessing the dominant maxima in the frequencies forming a geometrical progression, i.e. in the frequencies that are evenly distributed on a logarithmic scale. Examples of amplitudes and phases of the Fourier transform of base function $\psi_j(t)$ calculated from eq. (15) for $q = \sqrt{2}$ are presented in Fig. 2. An additional advantage of such a choice of base function $\psi_j(t)$ is the good conditioning of the least-squares problem matrix. This requires that the spectra of the functions $x(t)$ and $y(t)$ are not close to zero for frequencies corresponding to maxima of the amplitudes of Fourier transforms of $\psi_j(t)$. This is because when $i \neq j$ and $f(t)$ is arbitrary, the convolutions $f(t) * \psi_i(t)$ and $f(t) * \psi_j(t)$ contain other dominant frequencies and are linearly independent.

The discrete values of convolutions $\psi_j(t) * x(t)$ and $\psi_j(t) * y(t)$ that form the least-squares problem matrix are determined on the basis of eq. (8) as the difference of consecutively multiplied filtered sequences $\{x_i\}$ and $\{y_i\}$ through filters of three non-zero elements. To be more exact, the column of the matrix corresponding to an unknown α_j is determined from the formula

$$\begin{aligned}&\psi_j(t) * x(t) = u_j(t) - u_{j+1}(t), \qquad j = 0, \ldots, l-2;\\ &\psi_{l-1}(t) * x(t) = u_{l-1}(t),\end{aligned} \tag{11}$$

where $u_0(t) = h_0(t) * x(t) = x(t)$, $u_{j+1}(t) = u_j(t) * h_{j+1}(t)$ for $0 \le j < l-1$. We start the calculation of discrete values $u_{j,i} = u_j(t_0 + i\Delta t)$ of function $u_j(t)$ from the n-element sequence $\{x_i\}$. The difference between the most distant non-zero elements $h_j(i\Delta t)$ is $2k_j$, so the discrete-values sequence $\{u_{j,i}\}$ has $2k_j$ elements fewer than the sequence $\{u_{j-1,i}\}$. Thus, the sequence $\{u_{l-1,i}\}$ has $n - m_1 - 2\sum_{j=0}^{l-1} k_j$ elements and consequently the impulse response length $m = m_1 + m_2 = m_1 + 2\sum_{j=0}^{l-1} k_j$. In a similar manner we will define the function $w_j(t)$ for unknown β_j. Hence, taking into account eq. (8) we have the following relations:

$$\begin{aligned}&u_{0,i} = x_i, \quad w_{0,i} = y_i, \qquad i = 0, \ldots, n - m_1 - 1,\\ &u_{j+1,i} = \frac{1}{4}u_{j,i} + \frac{1}{2}u_{j,i-k_{j+1}} + \frac{1}{4}u_{j,i-2k_{j+1}},\\ &w_{j+1,i} = \frac{1}{4}w_{j,i} + \frac{1}{2}w_{j,i-k_{j+1}} + \frac{1}{4}w_{j,i-2k_{j+1}} \quad i = 2(k_0 + \cdots + k_{j+1}), \ldots, n - m_1 - 1, \quad j = 0, \ldots, l-2.\end{aligned} \tag{12}$$

Let vectors $\mathbf{u}_j$, $\mathbf{w}_j$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{x}_j$, $\mathbf{y}_j$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\mathbf{z}$ and $\mathbf{e}$ be defined by the equations

$$\mathbf{u}_j = [u_{j,m_2-1}, \ldots, u_{j,n-m_1-1}]^{\mathrm{T}},$$

$$\mathbf{w}_j = [w_{j,m_2-1}, \ldots, w_{j,n-m_1-1}]^{\mathrm{T}},$$

$$\boldsymbol{\alpha} = [\alpha_{l_p}, \ldots, \alpha_{l-1}]^{\mathrm{T}},$$

$$\boldsymbol{\beta} = [\beta_{l_p}, \ldots, \beta_{l-1}]^{\mathrm{T}},$$

$$\mathbf{x}_j = [x_{m_2-1-j}, \ldots, x_{n-m_1-1-j}]^{\mathrm{T}},$$

$$\mathbf{y}_j = [y_{m_2-1-j}, \ldots, y_{n-m_1-1-j}]^{\mathrm{T}},$$

$$\boldsymbol{\gamma} = [\gamma_{-m_1}, \ldots, \gamma_{m_3-1}]^{\mathrm{T}},$$

$$\boldsymbol{\delta} = [\delta_{-m_1}, \ldots, \delta_{m_3-1}]^{\mathrm{T}},$$

$$\mathbf{z} = [z_{m_2-1}, \ldots, z_{n-m_1-1}]^{\mathrm{T}},$$

$$\mathbf{e} = [1, \ldots, 1]^{\mathrm{T}},$$

form matrices

$$\mathbf{A}_\alpha = [\mathbf{u}_{l_p} - \mathbf{u}_{l_p+1}, \ldots, \mathbf{u}_{l-2} - \mathbf{u}_{l-1}, \mathbf{u}_{l-1}],$$

$$\mathbf{A}_\beta = [\mathbf{w}_{l_p} - \mathbf{w}_{l_p+1}, \ldots, \mathbf{w}_{l-2} - \mathbf{w}_{l-1}, \mathbf{w}_{l-1}],$$

$$\mathbf{A}_\gamma = [\mathbf{x}_{-m_1}, \ldots, \mathbf{x}_{m_3-1}],$$

$$\mathbf{A}_\delta = [\mathbf{y}_{-m_1}, \ldots, \mathbf{y}_{m_3-1}],$$

then the least-squares problem, $\min \|\varepsilon\|_2^2$, defined by eq. (6) can be written as follows

$$\min_{s,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}} \|[\mathbf{e}\ \mathbf{A}_\alpha\ \mathbf{A}_\beta\ \mathbf{A}_\gamma\ \mathbf{A}_\delta] \begin{bmatrix} s \\ \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \\ \boldsymbol{\delta} \end{bmatrix} - \mathbf{z}\|_2^2 = \min_{s,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}} \|\mathbf{A} \begin{bmatrix} s \\ \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \\ \boldsymbol{\delta} \end{bmatrix} - \mathbf{z}\|_2^2 . \tag{13}$$

The above problem can be solved using the orthogonal matrix triangularization (*QR* decomposition), which reduces the matrix to a simpler form. Householder orthogonalization is one of the methods for *QR* decomposition.

While using Householder's orthogonalization, the cost of the decomposition is $\sim(M - N/3)N^2$ arithmetic operations, the number of equations being $M = n - m + 1$, and the number of unknowns $N = 2l + 1 + m_1 + m_3$. Matrix $\mathbf{A}$ of the least-squares problem (13) has MN elements and, when the recordings are very lengthy, the matrix may be too large for the computer's operational memory. In such a case, it is enough to divide matrix $\mathbf{A}$ into matrices with a smaller number of rows and make a decomposition in stages. This is equivalent to the division of recordings into partly overlapping sections so that the number of observational equations without division of the recording is equal to the sum of equations for all the sections obtained from the division. In so doing, we can reduce the memory required by a factor equal to the number of parts into which the matrix $\mathbf{A}$ has been divided. Next steps in the *QR* decomposition of matrix $\mathbf{A}$ are made according to the following scheme. Assume that we have determined a triangular matrix $\mathbf{R}_j$ from the first j parts of matrix $\mathbf{A}$. Matrix $\mathbf{R}_{j+1}$ is determined from the *QR* decomposition of matrix $[\mathbf{A}_{j+1}^{\mathrm{T}}, \mathbf{R}_j^{\mathrm{T}}]^{\mathrm{T}}$.

The minimization problem (13) was written for a single data record. If we have more records, which may happen when records have gaps or were performed over different periods of time, we can make a generalization in an analogous manner to the above-mentioned case

of dividing the matrix **A** of eq. (13) into parts. The only difference is that the unknown s should be eliminated beforehand, since it may be different for different recordings (sections). Elimination of unknown s is made as follows: matrix **A** is multiplied by an orthogonal matrix **H** (Householder's) selected so that the first column of matrix **A** (corresponding to s) be transformed into a unit vector. Upon rejecting the first row and first column of matrix **HA**, we obtain matrix $\tilde{\mathbf{A}}$ of the least-squares problem without the unknown s.

From matrix **R**, we determine the unknowns α, β, γ and δ we are interested in. Errors of unknowns can be calculated in a standard way under the assumption that errors ε are uncorrelated and of the same accuracy. However, such an assumption is usually not fulfilled even in approximation: in the best although not very realistic situation of having the observation errors $\varepsilon_x(t)$, $\varepsilon_y(t)$ and $\varepsilon_z(t)$ of the measured functions $x(t)$, $y(t)$ and $z(t)$ uncorrelated and equally accurate, the minimized error

$$\varepsilon(t) = a(t) * \varepsilon_x(t) + b(t) * \varepsilon_y(t) - \varepsilon_z(t) \tag{14}$$

does not fulfil this property. That is why the calculated error estimate is to be treated as a crude indication only.

The determination of impulse responses is not the ultimate aim in the electromagnetic data processing. Of importance for us are the impedances and the geomagnetic transfer functions. The impulse responses, unlike the transfer function, are not uniquely defined in the sense that different functions in the time domain may have identical Fourier transforms over a finite interval in the frequency domain. Nevertheless, the impulse responses, in addition to serving for the determination of transfer functions, are very useful otherwise, namely for an easy determination of prediction, i.e. the function $a(t) * x(t) + b(t) * y(t)$. This is the simplest and most convenient method for assessing the quality of the recorded data and may be applied for their further selection.

The Fourier transforms $H_j(\omega)$ of function $h_j(t)$ and $\Psi_j(\omega)$ of function $\psi_j(t)$ determined from equations (8) and (9) are as follows

$$\begin{aligned} H_j(\omega) &= \frac{1}{2}(1 + \cos k_j \omega \Delta t)\mathrm{e}^{-ik_j\omega\Delta t}, \qquad j = 0, 1, \ldots, \\ \Psi_j(\omega) &= (1 - H_{j+1}(\omega))\prod_{i=0}^{j} H_i(\omega), \qquad j = 0, \ldots, l-2, \\ \Psi_{l-1}(\omega) &= \prod_{i=0}^{l-1} H_i(\omega). \end{aligned} \tag{15}$$

Hence, the transfer function $A(\omega)$ is expressed by the formula

$$A(\omega) = \sum_{j=l_p}^{l-1} \alpha_j \Psi_j(\omega) + \sum_{j=-m_1}^{m_3-1} \gamma_j \mathrm{e}^{-ij\omega\Delta t}. \tag{16}$$

$B(\omega)$ has a similar form.

Another property of the method should also be noted. In the least-squares problem (13) we took into account the observation equations corresponding to the whole sequence $\{z_i\}$. This is not necessary, and we can neglect part of the equations corresponding to those z_i which we know have been recorded with large errors (as a result of temporary noise or so) or not even recorded at all. However, we still must have at our disposal the complete sequences $\{x_i\}$ and $\{y_i\}$. From the practical point of view, this property of the method is very useful. While the magnetic components are less vulnerable to local noise and it is easy to keep the continuity of recording, the situation with electrical components is different. The signal-to-noise ratio for electrical components is often small, mainly because of local noise, and it is technically more difficult to obtain the continuous recording. Therefore, the fact that we can neglect some selected fragments of the electric field recordings in the computations is a great advantage.

We can even use the following procedure of automatic data selection. We determine the impulse response from the available data. On the basis of prediction we establish, depending on the number and quality of data, a certain number (say 10–30 per cent) of the worst-fitting electric field samples. We remove the corresponding observation equations and again determine impulse responses, this time more accurate ones. It is the level and type of noise that decides whether or not this procedure is reasonable. This procedure is essentially the same as that used in the frequency-domain robust processing where outliers are down-weighted or removed (Egbert 1997).

5 EXAMPLES OF COMPUTATIONS

The method described here has been tested on synthetic magnetotelluric data. We used one section containing 100 000 samples with a sample rate of 10 Hz. We used 2000 periods making up a geometrical sequence in the 0.3–4000 s range, with amplitudes and phases randomly generated with a uniform distribution (the amplitudes were additionally multiplied by T). As transfer functions, we used impedances corresponding to two one-dimensional models: a half-space of 10 Ω m resistivity and a one-layer model with a first, 6 km thick layer of 50 Ω m resistivity underlain by a 1 Ω m substratum. The results of calculations, apparent resistivities ρ_a and phases φ, are shown in Fig. 1. We adopted $q = 1.41$, $m_1 = 3$, $m_3 = 4$ and $l = 26$. The solid line represents the adopted model, the dashed line the calculated one.

In Fig. 3 we present the results of calculations for real data with the automatic data selection. The magnetotelluric recordings made at Belsk in the summer and autumn of 2001 and at the beginning of 2002 have been used. The solid line stands for the apparent resistivity and phase for a large number of observations (about 10^6 samples) from the summer and autumn of 2001 (10-s sampling interval). The dashed line was used for the results of calculations for the 2002 data (1.6×10^5 samples, 1 min sampling interval). The dashed line that strongly departs from the solid line represents the resistivity curve and phase calculated for all the data. In the 2002 electric data there are short-duration

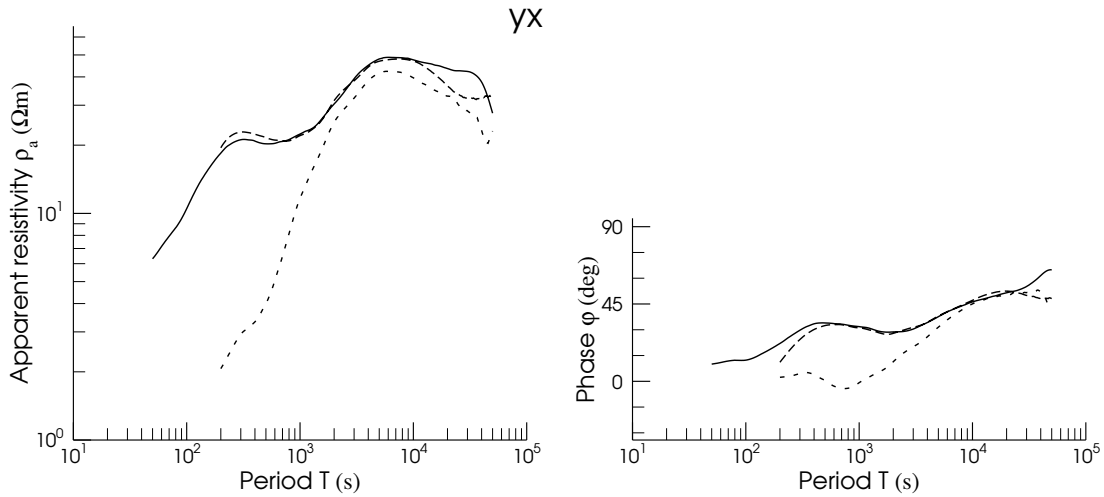

Figure 3. Apparent resistivity ρ_a and impedance phase φ calculated for the Belsk observatory. The solid line shows results obtained from data recorded in the summer 2001, and dashed lines present the results of 2002 data processing. The dashed line that strongly departs from the solid line is calculated from all data, and the other dashed line is calculated with automatic data selection.

large-amplitude noise signals that have not been eliminated, and which strongly affect the calculated impedance. After making three steps of selection and removing 20 per cent of the worst-fitting data, we obtained results similar to the 2001 results.

In Fig. 4 we show examples of the electric component prediction for the Belsk data for segments with (a) low and (b) high noise levels. Data disturbed by box-car shift noise (4b) have been identified and eliminated in consecutive iterations of impulse response calculations.

6 DISCUSSION

If the length of impulse response is fixed, the number of unknowns l to be determined depends upon coefficient q. If q decreases, the number of unknowns grows, and the approximation improves. However, in the case of real data, for which at many frequencies the signal-to-noise ratio may be small, the growing number of unknowns worsens the conditioning of the problem, and causes the determined transfer functions to be strongly oscillating. The tests performed suggest that for the real data $q = 2$ is sufficient. The maximum impulse response length is determined by both the duration of recording and the longest periods recorded. Unnecessary extension of the impulse response may also lead to incorrect results. To find an algorithm for proper selection of l and q on the basis of data is not an easy task. Therefore, it is necessary to make calculations for various values of l and q and then select the best solution on the basis of the determined impedances, predictions and possibly also the assessment of accuracy.

Close to the Nyquist frequency, it is only the unknowns $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ that are responsible for approximation. The choice of parameter q is not important, since even if q tends to 1, the maxima of function Ψ_j may be placed arbitrarily close to each other only provided that j is large enough, i.e. provided that frequencies are small enough. The tests show that it is enough to adopt $m_1 = m_3 = 3$ to ensure a good approximation to a frequency of $\pi/(1.5\Delta t)$. In order to get closer to the Nyquist frequency, it would be necessary to increase the number of unknowns δ_j and γ_j, and hence to worsen the conditioning of the problem.

The problems related to transfer function approximation close to the Nyquist frequency affects all the methods for determination of time-domain impulse response. In the frequency domain the transfer function in the Nyquist frequency always takes the real values ($\Im A(\pi/\Delta t) = 0$) and that is why it cannot be a good approximation of impedance, which is usually a complex number.

7 CONCLUSIONS

In the method presented here, the impulse responses are the linear combinations of appropriately defined base functions. Coefficients in these combinations are the unknowns to be determined by solving the linear least-squares problem. Numerical tests show that this approach makes it possible to reproduce exactly the transfer function for a broad range of frequencies. The major shortcomings of the proposed method are: (1) a worse approximation in the vicinity of the Nyquist frequency, a (2) simplified method of determining the errors of the solution, (3) less control over noise localized in frequency domain and (4) the need for a subjective choice of parameter q and the number of unknowns. We can consider the use of a regularization method to avoid the last disadvantage.

The main properties and advantages of the method are as follows:

(1) When there is a gap in the telluric component time-series, the algorithm skips this fragment of data.

(2) We have the possibility of making calculations in two steps. We start with all the data available, and then we reject, on the basis of prediction, those data that are contaminated with large errors.

(3) We can easily detect and remove from signals the noise that is localized in time.

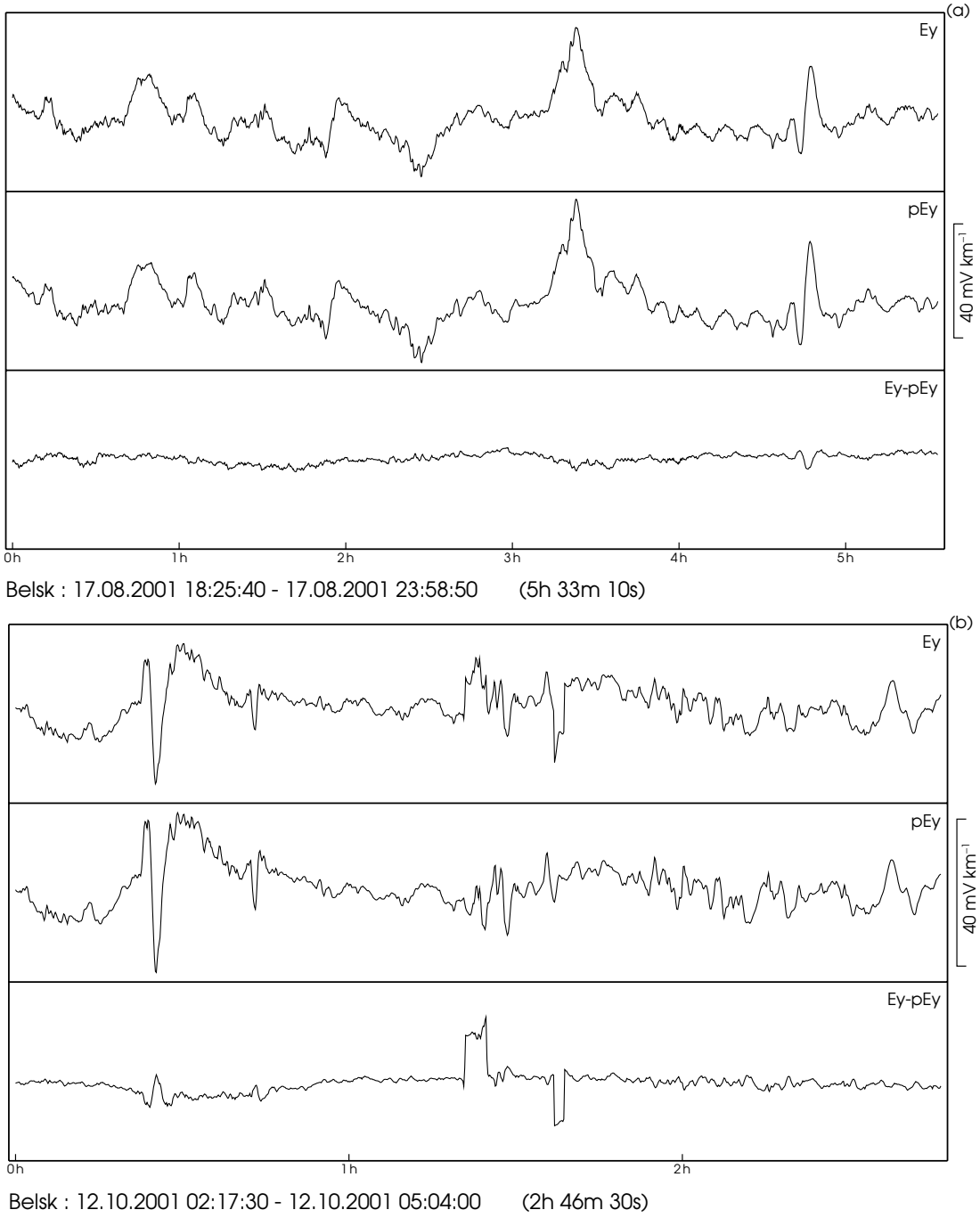

Figure 4. Examples of predictions (denoted pEx) for the Belsk data: (a) data with little noise, (b) data with box-car shift noise.

(4) We do not have to use a somewhat subjective procedure analogous to power spectra smoothing in the frequency domain.

(5) There are applications for the time-domain prediction beyond the improvement of MT impedances or geomagnetic transfer functions. For example, prediction can be used for removing natural source signals for *SP* studies or for finding non-inductive (external) part in vertical magnetic component H_z.

Treatment of gaps and multistep reweighted calculations are also possible for spectral approaches, but they are easier to implement in the time domain.

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