GEOPHYSICS, VOL. 72, NO. 1 (JANUARY-FEBRUARY 2007); P. R1–R7, 7 FIGS.
10.1190/1.2387117

A homotopy method for well-log constraint waveform inversion

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ABSTRACT

The primary difficulty posed by the method of waveform inversion for the acoustic-wave equation is the presence of local minima of the objective function. Waveform inversion fails to converge to the global minimum unless the wavefield contains very low frequencies, or the starting model is very close to the true model. Constraints can improve the convergence of the method; however, if the starting model is far from the correct model, normal iterative methods will still get trapped in local minima. We designed a homotopy method to improve the robustness of waveform inversion; it makes natural use of constraints, such as well logs. In addition, to further condition the inverse problem, we incorporate standard Tikhonov regularization. We demonstrate through a synthetic example that our method is more likely to find a global minimum than normal iterative methods.

INTRODUCTION

The problem of local minima is avoided in some of the earliest approaches to seismic inversion by linearizing the relation between the observed seismic data and the velocity model. The linearized inversion is justified when the initial velocity model is in the neighborhood of the global minimum of the objective function. This is a reasonable assumption for seismic inverse problems where the geology of the zone is not too complex or where good prior knowledge of the background velocity field is available. Numerous papers on linearized waveform inversion have been published (Berkhout, 1984; Devaney, 1984; Esmersoy, 1986; Levy and Esmersoy, 1988; Tarantola, 1984). However, these methods did not seem capable of recovering long wavelength components of the velocity model; consequently, they gave rise to questions about the observability of the complex wave-vector field in seismic data (see Bunks et al., 1995). Full nonlinear waveform velocity inversion as described in Mora (1987), Pica et al. (1990), and Tarantola (1986, 1988) seeks to determine the optimum velocity model by globally minimizing the objective function using an iterative descent method. But the direct application of this inversion technique to real and synthetic data has been disappointing because numerous local minima in the objective function prevent the iterative optimization techniques from working effectively. Scale decomposition has been very effective in dealing with this kind of problem, and striking results have been obtained by Kolb et al. (1986), Lindgren et al. (1989), Lindgren (1992), Saleck et al. (1993), Bunks et al. (1995), Meng and Liu (1998), Ghanem and Romeo (2001), Ascher and Haber (2003), and Fu and Han (2004).

A homotopy method is a powerful tool for solving nonlinear problems because of its property of global convergence (Watson, 1989). Homotopy methods should be useful for solving inverse problems in reflection seismology. Keller and Perozzi (1983) proposed a fast seismic-ray-tracing algorithm based on homotopy. Vasco (1994) then applied a homotopy method to the seismic traveltime tomography problem. The method was further developed and used to solving regularized inverse problems in seismic exploration (Vasco, 1998; Bube and Langan, 1999). Everett (1996) applied the method to solving the inverse electromagnetic (EM) problem based on finite difference modeling. Jegen et al. (2001) applied the homotopy method to the geophysical inversion of EM data based on the normal equations. Recently, Bao and Liu (2003) suggested a homotopy-regularization method to solve an inverse scattering problem. Feng et al. (2003) combined the homotopy method with the generalized pulse-spectrum technique to solve the inverse problem of the 2D wave equation. Zhang et al. (2004) studied homotopy methods for obtaining the impedance profile for the 1D wave equation.

In general, a waveform inversion identifies parameters using only seismograms recorded on the earth's surface, which may have a low signal-to-noise ratio. If wells are available, velocities from well logs, or seismograms from VSPs or other downhole data, may be less noisy than the surface data. So performing an inversion constrained by well logs might suppress the noise and improve the quality of inversion (see Wang et al., 1996).

Manuscript received by the Editor December 25, 2004; revised manuscript received August 18, 2006; published online November 15, 2006.
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A well-log constraint is introduced in this paper for the inversion of the 2D acoustic-wave equation, and a minimization problem is formed by using a finite-difference discretization. To overcome the ill-posedness of the problem, Tikhonov regularization is used. A widely convergent homotopy method is applied to the normal equations of the regularized functional to overcome the local convergence of traditional methods, and a practical method — a regularized homotopy method — is constructed. Numerical simulations conducted with a synthetic example indicate the effectiveness of the method proposed.

HOMOTOPY THEORY

Full waveform inversion of seismic data can be formulated as a nonlinear operator equation:

$$\mathbf{F}(\mathbf{m}) = \mathbf{y}, \tag{1}$$

where $\mathbf{m} = (m_1, m_2, \ldots, m_p)$ consists of p model parameters, and $\mathbf{y} = (y_1, y_2, \ldots, y_q)$ $(q \geq p)$ is a set of q observations. The model response, denoted by $\mathbf{F}(\mathbf{m})$, is computed by solving the appropriate wave equations that govern the forward problem. Equation 1 is usually highly nonlinear and severely ill-posed. Tikhonov regularization is certainly the most well-known regularization method (Tikhonov and Arsenin, 1977) for nonlinear ill-posed problems (Engl et al., 1989; Kaltenbacher, 1997). An alternative approach is to use iterative methods (e.g., Bakushinskii, 1992; Hanke et al., 1995; Hanke, 1997).

Indeed such iterative methods have good convergence properties under some restrictions; therefore, they have been widely used to deal with inverse problems like equation 1. However, all these methods are locally convergent. Good initial estimates or background velocities must be provided when these methods are used to solve operator equation 1. We will use the homotopy method to overcome this drawback. Next, we will briefly introduce the principle of homotopy theory for solving nonlinear operator equations like equation 1. For more details we refer to the tutorial provided by Watson (1989).

A homotopy function $\mathbf{H}(\mathbf{m}, \lambda)$ is constructed by adding to the "target function" $\mathbf{F}(\mathbf{m})$ a scalar homotopy parameter $\lambda \in [0,1]$ and a simple function $\mathbf{G}(\mathbf{m})$, so that

$$\mathbf{H}(\mathbf{m}, 0) = \mathbf{G}(\mathbf{m}),$$

$$\mathbf{H}(\mathbf{m}, 1) = \mathbf{F}(\mathbf{m}).$$

The starting system of the nonlinear equation $\mathbf{G}(\mathbf{m}) = 0$ can be chosen arbitrarily, but a known solution $\mathbf{m}^0$ should be chosen as the initial guess for the whole computation. If there is a curve $\mathbf{m} = \mathbf{m}(\lambda)$, $\lambda \in [0,1]$ satisfying the homotopy equation

$$\mathbf{H}(\mathbf{m}, \lambda) = 0, \tag{2}$$

it is then obvious that $\mathbf{m}(0) = \mathbf{m}^0$, and $\mathbf{m}(1) = \mathbf{m}^*$, which is a solution of operator equation 1. Therefore $\mathbf{m}(\lambda)$ is a curve connecting $\mathbf{m}^0$ with $\mathbf{m}^*$. Using some numerical method to trace this curve from a known point $\mathbf{m}^0$, we hope to reach the solution $\mathbf{m}^*$. Basically, there are two ways to trace this curve. The first is based on the assumption that $\mathbf{H}(\mathbf{m}, \lambda)$ is differentiable with respect to $\mathbf{m}$ and λ. By differentiating each side of equation 2 on λ,

$$\mathbf{H}'_{\mathbf{m}}(\mathbf{m}, \lambda) d\mathbf{m} + \mathbf{H}'_{\lambda}(\mathbf{m}, \lambda) d\lambda = 0. \tag{3}$$

Since $\mathbf{m}$ satisfies the initial condition $\mathbf{m}(0) = \mathbf{m}^0$, determining $\mathbf{m}(\lambda)$ turns out to be an initial value problem of ordinary differential equation 3. By using numerical integration, one can obtain an approximation of $\mathbf{m}^* = \mathbf{m}(1)$.

The second method is to divide the interval $[0,1]$ into $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N = 1$, then use iterative methods to solve the operator equation sequentially:

$$\mathbf{H}(\mathbf{m}, \lambda_k) = 0, \quad k = 1, 2, \ldots, N. \tag{4}$$

By induction, since the solution $\mathbf{m}^{k-1}$ is known, $\mathbf{m}^{k-1}$ can be used as the initial approximation of the kth equation of equation 4. When $\lambda_k - \lambda_{k-1}$ is small enough, $\mathbf{m}^{k-1}$ would be expected to be a good approximation to $\mathbf{m}^k$, such that we only need a locally convergent method to reach $\mathbf{m}^k$. In this paper, we use the latter method to trace the path $m(\lambda)$.

INVERSE MODEL

The equation for acoustic waves in 2D is described by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2(x,z)} \frac{\partial^2 u}{\partial t^2} = g(t)\delta(z), \quad t > 0, \tag{5}$$

where $\delta(\cdot)$ is Dirac δ-function, $v(x,z)$ is the velocity at (x,z) in the medium, $g(t)$ is a source time function such that $g(t) = 0$, $t < 0$, and the source is located at $(0,0)$ on the surface of the earth. The horizontal and vertical coordinates are x, z, respectively, and t is the coordinate in time. The surface of the earth corresponds to $z = 0$. For simplicity, the problem is studied in the domain $\Omega = [0,L] \times [0,H]$, under boundary conditions

$$\left.\frac{\partial u(x,z,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial u(x,z,t)}{\partial x}\right|_{x=L} = 0, \tag{6}$$

$$\left.\frac{\partial u(x,z,t)}{\partial z}\right|_{z=H} = 0, \tag{7}$$

and initial conditions

$$u(x,z,0) = \frac{\partial u(x,z,0)}{\partial t} = 0. \tag{8}$$

Equations 5–8 form the forward problem of the acoustic-wave equation, but in practice, velocity v is not known. What we know is based on observed wavefield data, e.g.,

$$u(x,0,t) = f(x,t), x \in [0,L], t \in [0,T]. \tag{9}$$

In this case, the receivers are located on the surface of the earth. We can reconstruct the velocity from equations 5–8 and the observed data f. Wave velocities $v(x_0,z)$ at all depths of constraint point x_0 are necessary for constrained inversion, and they can be obtained from the seismograms of well logs.

DISCRETIZATION

Equations 5–8 can be approximated by the following second-order, finite-difference equations:

$$u_{i,j}^{k+1} = \frac{\tau^2}{h_x^2} v_{i,j}^2 (u_{i-1,j}^k + u_{i+1,j}^k - 2u_{i,j}^k) + \frac{\tau^2}{h_z^2} v_{i,j}^2 (u_{i,j+1}^k + u_{i,j-1}^k - 2u_{i,j}^k) + 2u_{i,j}^k - u_{i,j}^{k-1},$$

$$
\begin{aligned}
& i = 1,2, \ldots ,I-1; \quad j = 1,2, \ldots ,J-1 \\
u_{i,0}^{k+1} &= \frac{\tau^2}{h_x^2} v_{i,0}^2 (u_{i-1,0}^k + u_{i+1,0}^k - 2u_{i,0}^k) \\
&\quad + \frac{\tau^2}{h_z^2} v_{i,0}^2 (u_{i,1}^k - 2u_{i,0}^k) + 2u_{i,0}^{k-1} \\
&\quad - u_{i,0}^{k-1} - g^k \tau^2 v_{i,0}^2, \quad i = 1,2, \ldots ,I-1, \\
u_{0,j}^k &= u_{1,j}^k, u_{I,j}^k = u_{I-1,j}^k, \quad j = 0,1, \ldots ,J, \\
u_{i,J}^k &= u_{i,J-1}^k, \quad i = 0,1, \ldots ,I, \\
u_{i,j}^0 &= u_{i,j}^1 = 0, i = 0,1, \ldots ,I; \quad j = 0,1, \ldots ,J, \\
u_{i,0}^k &= f_i^k, \quad i = 0,1, \ldots ,I,
\end{aligned} \tag{10}
$$

where $u_{i,j}^k = u(i \times h_x, j \times h_z, k \times \tau)$, h_x, h_z are the step sizes of the rectangular grid in the x- and z-directions, respectively, and τ is the time step length, $I = \frac{L}{h_x}, J = \frac{H}{h_z}, v_{i,j} = v(ih_x, jh_z)$.

Difference equation 10 defines a nonlinear operator equation $\mathbf{F}(\mathbf{m}) = \mathbf{y}$, where $\boldsymbol{m}$ denotes a vector that is composed of v_{ij} in a suitable sequence, and $\mathbf{y}$ denotes a vector that is composed of f_i^k in a suitable sequence. Let $\bar{f}_i^k$ denote the observed data and form vector $\bar{\boldsymbol{y}}$ in the same sequence as $\mathbf{y}$. Especially, let

$$
\begin{aligned}
\mathbf{m}^T &= (v_{1,0}, v_{1,1}, \ldots, v_{1,J-1}, \ldots, v_{I-1,0}, v_{I-1,1}, \ldots, v_{I-1,J-1}), \\
\mathbf{y}^T &= (f_1^1, f_2^1, \ldots, f_{I-1}^1, f_1^2, f_2^2, \ldots, f_{I-1}^2, \ldots, f_1^l, f_2^l, \ldots, f_{I-1}^l), \\
\bar{\mathbf{y}}^T &= (\bar{f}_1^1, \bar{f}_2^1, \ldots, \bar{f}_{I-1}^1, \bar{f}_1^2, \bar{f}_2^2, \ldots, \bar{f}_{I-1}^2, \ldots, \bar{f}_1^l, \bar{f}_2^l, \ldots, \bar{f}_{I-1}^l).
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathbf{m}_i^T &= (v_{i,0}, v_{i,1}, v_{i,2}, \ldots, v_{i,J-1}), \\
\bar{\mathbf{m}}_{i_0}^T &= (\bar{v}_{i_0,0}, \bar{v}_{i_0,1}, \bar{v}_{i_0,2}, \ldots, \bar{v}_{i_0,J-1}),
\end{aligned}
$$

where $\bar{\mathbf{m}}_{i_0}^T$ is the velocity from the well logs of a well located at point i_0 in x, $0 < i_0 < I$, and T denotes the transpose of a vector. By defining the admissible set

$$\boldsymbol{\Sigma} = (\mathbf{m}, \mathbf{m}_{i_0} = \bar{\mathbf{m}}_{i_0}),$$

the inversion of $\mathbf{m}$ can be transformed into the solution of the problem

$$\min_{\mathbf{m} \in \boldsymbol{\Sigma}} \|\mathbf{F}(\mathbf{m}) - \bar{\mathbf{y}}\|^2, \tag{11}$$

where $\|\cdot\|$ is the l^2-norm. Because this problem is difficult to solve, we transform it into another easier-to-solve form.

First, we assume

$$\mathbf{m}^T = (\underbrace{0,0, \ldots ,0}_{I \times (i_0 - 1)}, \bar{v}_{i_0,0}, \bar{v}_{i_0,1}, \bar{v}_{i_0,J-1}, 0,0, \ldots ,0)$$

$$
\mathbf{D} = \begin{bmatrix}
0 & & & & & & & & & & & \\
& 0 & & & & & & & & & & \\
& & \ddots & & & & & & & & & \\
& & & 0 & & & & & & & & \\
& & & & 1 & & & & & & & \\
& & & & & 1 & & & & & & \\
& & & & & & \ddots & & & & & \\
& & & & & & & 1 & & & & \\
& & & & & & & & 0 & & & \\
& & & & & & & & & \ddots & & \\
& & & & & & & & & & 0
\end{bmatrix}.
$$

Obviously, $\|\mathbf{Dm} - \bar{\mathbf{m}}\| = 0$ holds for $\mathbf{m} \in \boldsymbol{\Sigma}$. Thus, minimization equation 11 can be written, without constraint, as

$$\min_{\mathbf{m} \in \boldsymbol{\Sigma}} [\|\mathbf{F}(\mathbf{m}) - \bar{\mathbf{y}}\|^2 + \alpha_3 \|\mathbf{Dm} - \bar{\mathbf{m}}\|^2], \tag{12}$$

where α_3 is a constraint parameter to determine the strength of the constraint. If α_3 is large enough, the minima of equations 11 and 12 are close to each other. Thus in the specific inversion process, α_3 must be specified large enough so that the minimum of equation 12 can approximate that of equation 11 well.

INVERSION METHOD

It is well known that the inverse problem in equation 12 is ill-posed. To improve the numerical stability of the algorithm, a regularization technique should be introduced:

$$
\begin{aligned}
\min_{\mathbf{m} \in \boldsymbol{\Sigma}} [&\|\mathbf{F}(\mathbf{m}) - \bar{\mathbf{y}}\|^2 + \alpha_1 \|\mathbf{M}_1(\mathbf{m} - \mathbf{m}^0)\|^2 \\
&+ \alpha_2 \|\mathbf{M}_2(\mathbf{m} - \mathbf{m}^0)\|^2 + \alpha_3 \|\mathbf{Dm} - \bar{\mathbf{m}}\|^2],
\end{aligned} \tag{13}
$$

where $\mathbf{M}_1, \mathbf{M}_2$ are the second-order smoothing matrices in the x- and z-directions, respectively (see Wang et al., 1996), and α_1 and α_2 are the regularization parameters.

It is easy to see that the minimum $\mathbf{m}^*$ of equation 13 satisfies the normal Euler equation

$$
\begin{aligned}
\mathbf{F}'(\mathbf{m})^T(\mathbf{F}(\mathbf{m}) - \bar{\mathbf{y}}) &+ (\alpha_1 \mathbf{M}_1^T \mathbf{M} + \alpha_2 \mathbf{M}_2^T \mathbf{M}_2)(\mathbf{m} - \mathbf{m}^0) \\
&+ \alpha_3 \mathbf{D}^T(\mathbf{Dm} - \bar{\mathbf{m}}) = 0.
\end{aligned} \tag{14}
$$

In order to solve equation 14, a successive linearization method is used to construct a basic iterative method, because a second-order derivative may appear if the Newton iterative method is used.

Supposing that the kth approximation $\mathbf{m}^k$ of $\mathbf{m}^*$ has already been found, $\mathbf{F}(\mathbf{m})$ is replaced by the linear function $\mathbf{L}_k(\mathbf{m}) = \mathbf{F}'(\mathbf{m}^k)(\mathbf{m} - \mathbf{m}^k) + \mathbf{F}(\mathbf{m}^k)$, and the objective function in equation 13 by

$$
\begin{aligned}
\mathbf{J}_k(\mathbf{m}) = \|\mathbf{L}_k(\mathbf{m}) - \bar{\mathbf{y}}\|^2 &+ \alpha_1 \|\mathbf{M}_1(\mathbf{m} - \mathbf{m}^k)\|^2 \\
&+ \alpha_2 \|\mathbf{M}_2(\mathbf{m} - \mathbf{m}^k)\|^2 + \alpha_3 \|\mathbf{Dm} - \bar{\mathbf{m}}\|^2
\end{aligned}
$$

for computing the next approximation $\mathbf{m}^{k+1}$. The corresponding Euler equation to the minimization problem $\min_{\mathbf{m}\in\Sigma} \mathbf{J}_k(\mathbf{m})$ is

$$\mathbf{F}'(\mathbf{m}^k)^T[\mathbf{F}'(\mathbf{m}^k)(\mathbf{m}-\mathbf{m}^k)+\mathbf{F}(\mathbf{m}^k)-\bar{\mathbf{y}}]+(\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2)(\mathbf{m}-\mathbf{m}^k)+\alpha_3\mathbf{D}^T(\mathbf{D}\mathbf{m}-\bar{\mathbf{m}})=0. \tag{15}$$

The solution of equation 15 can be regarded as the next approximation $\mathbf{m}^{k+1}$ to $\mathbf{m}^*$:

$$\mathbf{m}^{k+1}=\mathbf{m}^k-[\mathbf{F}'(\mathbf{m}^k)^T\mathbf{F}'(\mathbf{m}^k)+\alpha_1\mathbf{M}_1^T\mathbf{M}_1+\alpha_2\mathbf{M}_2^T\mathbf{M}_2 + \alpha_3\mathbf{D}^T\mathbf{D}]^{-1}\cdot[\mathbf{F}'(\mathbf{m}^k)^T(\mathbf{F}(\mathbf{m}^k)-\bar{\mathbf{y}}) + \alpha_3\mathbf{D}^T(\mathbf{D}\mathbf{m}^k-\bar{\mathbf{m}})],\quad k=0,1,\ldots. \tag{16}$$

This iterative method should converge rapidly, but it is only locally convergent as are most iterative methods. In fact, equation 16 is a variant of the iteratively regularized Gauss-Newton method (see Bakushinskii, 1992), and the computational cost is nearly the same as the latter. In order to overcome purely local convergence, we use fixed-point homotopy equation 2 to solve equation 14:

$$\mathbf{H}(\mathbf{m},\lambda)=\lambda[\mathbf{F}'(\mathbf{m})^T(\mathbf{F}(\mathbf{m})-\bar{\mathbf{y}})+(\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2)(\mathbf{m}-\mathbf{m}^0)+\alpha_3\mathbf{D}^T(\mathbf{D}\mathbf{m}-\bar{\mathbf{m}})] + (1-\lambda)(\mathbf{m}-\mathbf{m}^0)=0.$$

Here, the simple map is $\mathbf{G}(\mathbf{m})=\mathbf{m}-\mathbf{m}^0$, and the target function is the left-hand side of equation 14. The equation above can also be rearranged as

$$\lambda[\mathbf{F}'(\mathbf{m})^T(\mathbf{F}(\mathbf{m})-\bar{\mathbf{y}})+\alpha_3\mathbf{D}^T(\mathbf{D}m-\bar{\mathbf{m}})]+[\lambda(\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2)+(1-\lambda)\mathbf{I}](\mathbf{m}-\mathbf{m}^0)=0. \tag{17}$$

To obtain $\mathbf{m}^*$, interval $[0,1]$ can be divided into $0=\lambda_0<\lambda_1<\cdots<\lambda_N=1$. For $\lambda=\lambda_k$, we use an iterative method to solve equation 17 sequentially. As the solution $\mathbf{m}^0$ of equation 17 is known when $\lambda=0$, $\mathbf{m}^0$ can be used as the initial approximation of the next equation. Assuming that the solution $\mathbf{m}^k$ of the kth equation has already been obtained, the successive linearization method can be used to solve the $(k+1)$th equation, which is done with equation 16. We then have the iteration formula

$$\begin{aligned}\mathbf{m}_{j+1}^k=\mathbf{m}_j^k-&[\lambda_k\mathbf{F}'(\mathbf{m}_j^k)^T\mathbf{F}'(\mathbf{m}_j^k)+\lambda_k(\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2)+(1-\lambda_k)\mathbf{I} + \alpha_3\lambda_k\mathbf{D}^T\mathbf{D}]^{-1}\cdot\{\lambda_k\mathbf{F}'(\mathbf{m}_j^k)^T[\mathbf{F}(\mathbf{m}_j^k)-\bar{\mathbf{y}}]\\ &+[\lambda_k(\alpha_1\mathbf{M}_1^T\mathbf{M}_1+\alpha_2\mathbf{M}_2^T\mathbf{M}_2)+(1-\lambda_k)\mathbf{I}]\times(\mathbf{m}_j^k-\mathbf{m}^0)+\lambda_k\alpha_3\mathbf{D}^T(\mathbf{D}\mathbf{m}_j^k-\bar{\mathbf{m}})\},\\ &j=0,1,\ldots,k_T,\mathbf{m}^{k+1}=\mathbf{m}_{k_T+1}^k,k=0,1,\ldots,N-1.\end{aligned} \tag{18}$$

The approximation $\mathbf{m}^N$ of the Nth equation as the initial guess of equation 16 can be solved with equation 18. Iteration is repeated until the regularized approximation of $\mathbf{m}^*$ is obtained. Therefore, equations 16 and 18 form a stabilized method that should have a wider domain of convergence for the constrained inversion of the 2D acoustic-wave equation.

If $\lambda_k=k/N$ by an isometric division, and $k_T=0$, equation 18 is then simplified as follows:

$$\begin{aligned}\mathbf{m}^{k+1}=\mathbf{m}^k-&\left[\frac{k}{N}\mathbf{F}'(\mathbf{m}^k)^T\mathbf{F}'(\mathbf{m}^k)+\frac{k}{N}(\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2)+\left(1-\frac{k}{N}\right)\mathbf{I}+\frac{k}{N}\alpha_3\mathbf{D}^T\mathbf{D}\right]^{-1}\\ &\cdot\left\{\frac{k}{N}\mathbf{F}'(\mathbf{m}^k)^T[\mathbf{F}(\mathbf{m}_j^k)-\bar{\mathbf{y}}]+\left[\frac{k}{N}(\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2)+\left(1-\frac{k}{N}\right)\mathbf{I}\right]\times(\mathbf{m}^k-\mathbf{m}^0)+\frac{k}{N}\alpha_3\mathbf{D}^T(\mathbf{D}\mathbf{m}^k-\bar{\mathbf{m}})\right\},\\ &k=0,1,\ldots,N-1.\end{aligned} \tag{19}$$

If $\alpha_3=0$, equation 13 is then the ordinary inverse problem of the 2D acoustic wave equation, and equations 16 and 19 can be expressed as

$$\mathbf{m}^{k+1}=\mathbf{m}^k-[\mathbf{F}'(\mathbf{m}^k)^T\mathbf{F}'(\mathbf{m}^k)+\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2]^{-1}\cdot[\mathbf{F}'(\mathbf{m}^k)^T(\mathbf{F}(\mathbf{m}^k)-\bar{\mathbf{y}})],\ k=0,1,\ldots \tag{20}$$

and

$$\begin{aligned}\mathbf{m}^{k+1}=\mathbf{m}^k-&\left[\frac{k}{N}\mathbf{F}'(\mathbf{m}^k)^T\mathbf{F}'(\mathbf{m}^k)+\frac{k}{N}(\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2)+\left(1-\frac{k}{N}\right)\mathbf{I}\right]^{-1}\\ &\cdot\left\{\frac{k}{N}\mathbf{F}'(\mathbf{m}^k)^T[\mathbf{F}(\mathbf{m}^k)-\bar{\mathbf{y}}]+\left[\frac{k}{N}(\alpha_1\mathbf{M}_1^T\mathbf{M}_1 + \alpha_2\mathbf{M}_2^T\mathbf{M}_2)+\left(1-\frac{k}{N}\right)\mathbf{I}\right](\mathbf{m}^k-\mathbf{m}^0)\right\},\\ &k=0,1,\ldots,N-1.\end{aligned} \tag{21}$$

In comparing the proposed method with other well-known iterative methods, such as the regularized Gauss-Newton method, the fundamental computations are nearly the same at each step. Primarily, these are the application of the first-order derivative and its adjoint, operator evaluation, and a forward-modeling run. Storage requirements are also similar to the Gauss-Newton method. Most importantly; however, the region of convergence of the proposed method can be much broader than iterative techniques like the Gauss-Newton method.

NUMERICAL TEST RESULTS AND DISCUSSION

In this section, a synthetic example is tested to show the performance of our inversion algorithm. With the source as shown in Figure 1, equispaced receivers on the surface along the x-axis (one receiver for each grid point in the x-direction) and the exact velocity model are shown in Figure 2. The seismograms are generated from the original velocity model using the finite-difference operator described in the section on discretization and equation 9 at every grid point. This experiment is a surface-reflection survey. The purpose of this test is to demonstrate that our algorithm can effectively avoid being trapped in local minima of the inverse problem and improve the robustness of waveform inversion.

In this numerical test, parameter N in equations 19 and 21 is chosen to be 10. For $h_x = 10$ m, $h_z = 5$ m, and $\tau = 0.01$ s there are 30 by 60 grid points, and the seismograms obtained from the forward computation are as shown in Figure 3. In every numerical test, iteration continues until the relative error is equal to or less than 2%. A well with known velocities is assumed to be located at the 15th grid point in the horizontal direction, i.e., $i_0 = 15$ where the velocity is known. Initial value $v^0(x,z)$ is selected as 2800 m/s for all the numerical tests. When there is no noise, as in the constrained inversion results shown in Figure 4, the inversion result of the eighth iteration of equation 16 after $\mathbf{m}^{10}$ is obtained by equation 19. In this numerical test, parameters α_1, α_2, and α_3 are 5×10^{-2}, 5×10^{-2}, and 1×10^4, respectively. In order to test the noise suppression ability of the proposed method, 10% random noise is added to the seismograms, as shown in Figure 5. For such seismograms, numerical simulation is carried out, and the result is shown in Figure 6. In this numerical test, parameters α_1, α_2, and α_3 are 2, 2, and 1×10^4, respectively. To test the necessity of the introduction of the well logs, we invert the velocity for the same noisy seismograms without using the well data. The inversion result is shown in Figure 7. Compared to Figure 6, it is obvious that the result shown in Figure 7 is less robust.

It can be seen above that when there is no noise, eight iterations of the basic method in equation 16 are needed; thus, the computational speed is relatively rapid and the accuracy/robustness high. When noise is added, accuracy decreases. With 10% noise, the inversion results still have very high accuracy. With 15% noise added, the result is still satisfactory for the well-log-constrained inversion, but we do not obtain an acceptable result when there is no well log for such

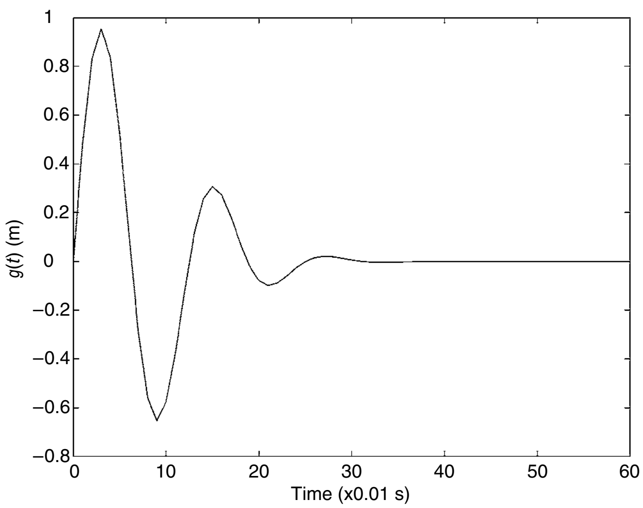

Figure 1. The source function.

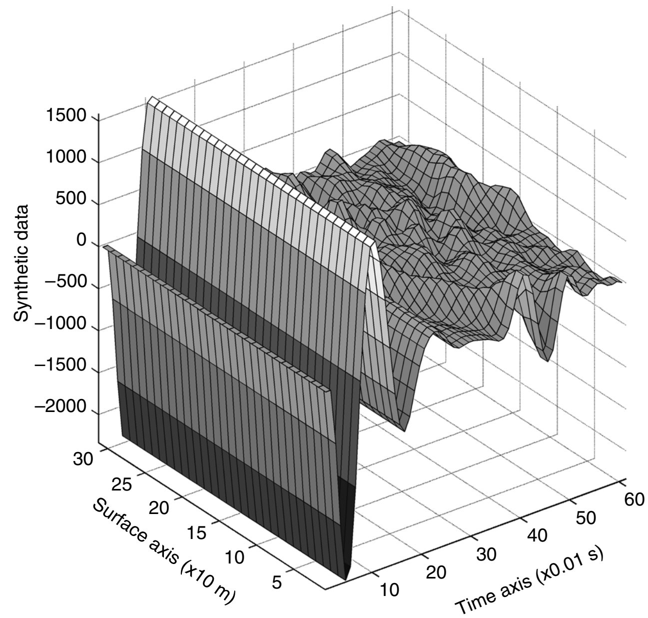

Figure 3. The synthetic data calculated from the forward computation.

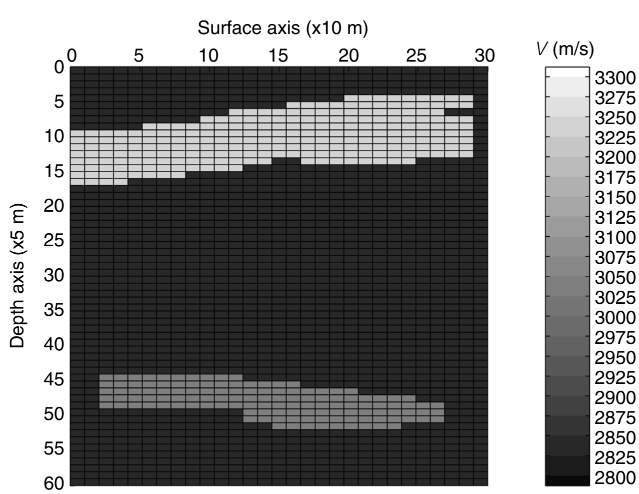

Figure 2. The true velocity model. The model is 300×300 m, and there are 30 grid points in the x-direction and 60 in the z-direction.

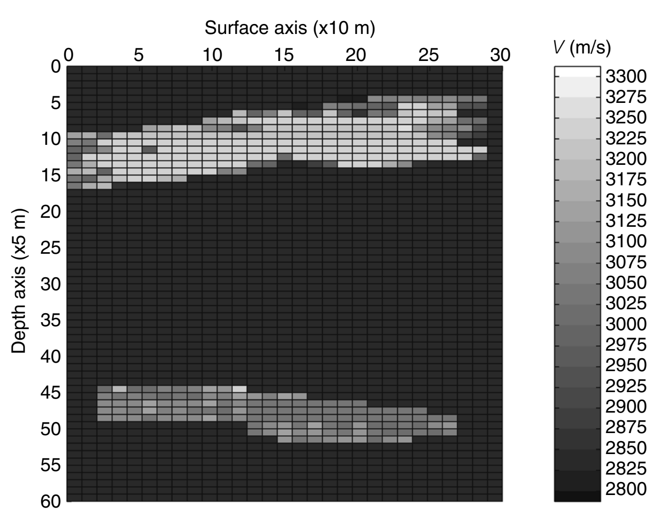

Figure 4. The result of the well-log constrained inversion for the noise-free data in Figure 3.

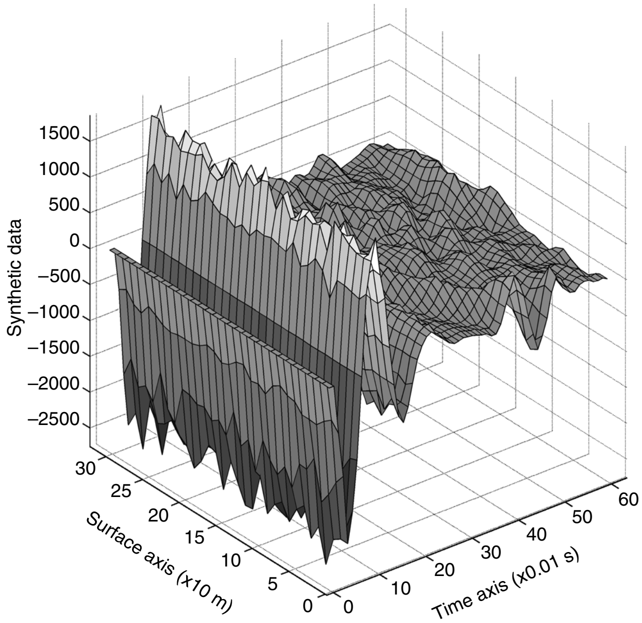

Figure 5. The noisy synthetic data obtained by adding 10% random noise to the synthetic data shown in Figure 3.

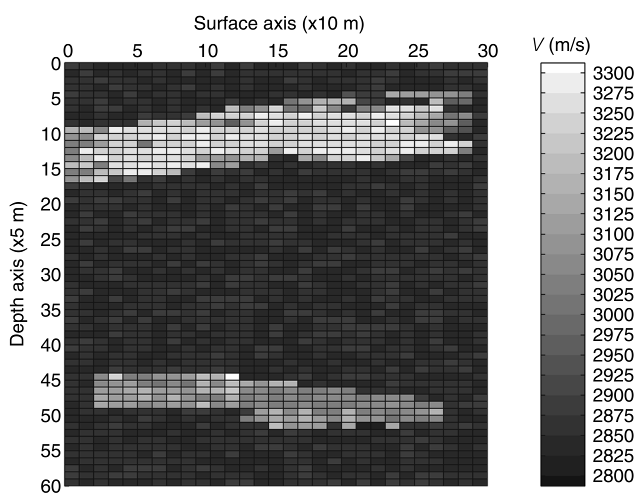

Figure 6. The result of the well-log constrained inversion using the noisy synthetic data shown in Figure 5, compared to Figure 2.

noisy data. This shows the necessity of the constrained inversion. To demonstrate the ability of the proposed method equations 19 and 16 to converge from a distant initial guess compared to other methods, we applied equation 16 alone and found that the direct application of that iterative technique will diverge if iterations start directly from $v^0(x,z) \equiv 2800$ m/s. This indicates that the proposed method is more likely to find a global minimum. In this numerical test, the velocities in the entire field had to be estimated without a good initial model. If the initial model were chosen even farther from the true model, the method would still converge, but the number of intervals N might have to be increased.

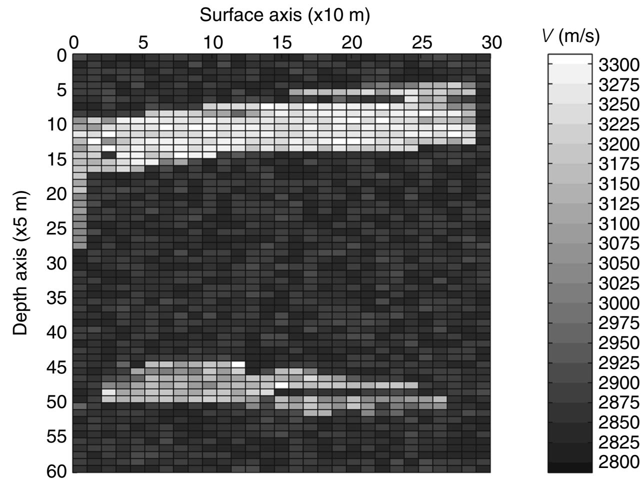

Figure 7. The inversion result using the noisy synthetic data shown in Figure 5 without well data, compared to Figure 6.

CONCLUSIONS

In order to improve the resolution of inversion, a well-log constraint is introduced to the 2D wave equation inversion. Then Tikhonov regularization is applied to the minimization problem obtained by the discretization of the inverse problem to overcome the ill-posedness. We constructed a homotopy method that further widens the domain of convergence of traditional methods. We conclude from the numerical test results that the noise suppression ability, the stability, and the resolution of the results are greatly improved compared to standard iterative methods. Because all geophysical inverse problems have similar problems of low resolution, ill-posedness, and nonlinearity, we believe that the method proposed here will have a significant effect on the development of practical solutions to large-scale geophysical inverse problems.

ACKNOWLEDGMENTS

This work is funded by the National Science Foundation of China under grant no. 40544016. We thank Yonghe Sun and the anonymous reviewers for their comments and suggestions.

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