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The Effect of the Intermediate Principal Stress on the Ground Response of Circular Openings in Rock Mass

By

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List of Symbols

b	strength criterion coefficient in the unified strength criterion
c	cohesive strength
G	shear modulus
m	compressive-tensile strength ratio
p_i, p_i^{cr}, p_o	internal pressure, critical internal pressure and external pressure
p_c	radial stress at the elasto-plastic interface
r_i, r_o	internal radius and external radius
r_c	extent of the plastic zone
u, u_i	radial displacement and its value at internal wall of the opening
β	material parameter in the unified strength criterion
ε_r, ε_θ	radial and tangential strains
φ	friction angle
ν	Poisson's ratio
σ_1, σ_2, σ_3	principal stresses
σ_{13}, σ_{12}, σ_{23}	normal stresses defined as $\sigma_{13} = (\sigma_1 + \sigma_3)/2$, $\sigma_{12} = (\sigma_1 + \sigma_2)/2$, $\sigma_{23} = (\sigma_2 + \sigma_3)/2$
σ_c	uniaxial compressive strength
σ_r, σ_θ, σ_z	radial, tangential and axial stresses
τ_{13}, τ_{12}, τ_{23}	principal shear stresses, i.e., $\tau_{13} = (\sigma_1 - \sigma_3)/2$, $\tau_{12} = (\sigma_1 - \sigma_2)/2$, $\tau_{23} = (\sigma_2 - \sigma_3)/2$

1. Introduction

The opening problem of an elasto-plastic geomaterial is of interest to geotechnical engineers for stability and closure predictions in tunneling, to petroleum engineers for borehole drilling, and to mining engineers for shaft design. Hence, accurate prediction of the ground response of the opening is of importance for understanding the problem and for the support design in geotechnical engineering. To our knowledge, two analytical approaches can be applied to deal with this issue. One is by using theory of continuum damage mechanics such as given by Wang (1992a, 1992b, 1993). Another is by using theory of plasticity to obtain the solution for stresses and displacement around the opening. The current work is focused on the latter approach.

As early as 1938, Fenner obtained a formula for the extent of the plastic zone of a circular opening in elasto-perfectly plastic geomaterial subjected to uniform internal and far-field stresses on the plane strain condition and Mohr-Coulomb criterion. Fenner's formula ignores cohesion of the material at the elasto-plastic interface, and was improved by Kastner (1962). Subsequently, Salencon presented a more systematic elasto-plastic analysis to the same problem (1969). In recent two decades, many authors made efforts to extend the work by Fenner, Kastner and Salencon (Kennedy and Lindberg, 1978; Brown et al., 1983; Reed, 1986; Detournay, 1986; Detournay and Fairhurst, 1987; Ogawa and Lo, 1987; Wang, 1994, 1996; Papanastasiou and Durban, 1997; Carranza-Torres and Fairhurst, 1999; Chen et al., 1999; Jiang et al., 2001; Sharan, 2003; Carranza-Torres, 2003; Alonso et al., 2003).

The above studies are based on Mohr-Coulomb criterion, or Hoek-Brown criterion, or Drucker-Prager criterion, and cannot consider the intermediate principal stress effect of the rock mass reasonably, for both Mohr-Coulomb criterion and Hoek-Brown criterion do not include the intermediate principal stress effect at all, and Drucker-Prager criterion includes this effect but equates it to the effect of the maximum principal stress and the minimum principal stress. In fact, the strength of geomaterials (such as soil, rock mass, etc.) is often observed to be dependent on the intermediate principal stress, and moreover, this effect varies from case to case and the extent of the effect is related to the material type and the stress state (Shibata and Karube, 1965; Mogi, 1967; Michelis, 1985; Singh et al., 1998). In addition, the aforementioned studies, except those by Wang (1994, 1996) and Chen et al. (1999), are all restricted to openings in an infinite medium, which may be inapplicable for a hollow cylinder test.

As for a strength criterion considering all the three principal stresses more reasonably, Yu and He (1991) proposed a unified strength criterion (USC), which has a piecewise linear expression, distinct physical meaning. The USC has been applied successfully in some cases since its initiation (e.g., Ma et al., 1999; Yu et al., 2002; Yu, 2002, 2004; Zhang et al., 2003). In this paper, the USC is used to describe the behavior of the rock mass in the plastic range in the opening problems. Under plane strain conditions and on the basis of theories of elasticity and ideal plasticity, the closed-form solutions are presented for the stresses, displacement and the extent of the plastic zone of an opening with a finite external radius in a rock mass subjected to a uniform internal pressure and external pressure. The present solutions have capability to take account of the intermediate principal stress effect of rock mass, and can be applied to both a hollow cylinder test in the laboratory and an opening in an infinite medium. It is shown that Fenner's formula and Kastner's formula are all special cases of the present solutions. Finally, the influence of the intermediate principal stress on the stress distributions, the wall displacement, the extent of the plastic zone and the ground response curve of a tunnel are investigated by using the present solutions.

2. Review of the Unified Strength Criterion

Based on orthogonal octahedron of the twin-shear element model (Yu et al., 1985), the USC specifies that the failure of material be decided by the two larger principal shear stresses and their corresponding normal stresses on the orthogonal octahedron element. The mathematical expression of the USC is as follows (Yu and He, 1991)

$$\tau_{13} + b\tau_{12} + \beta(\sigma_{13} + b\sigma_{12}) = C \quad (\tau_{12} + \beta\sigma_{12} \geqslant \tau_{23} + \beta\sigma_{23}), \tag{1a}$$

$$\tau_{13} + b\tau_{23} + \beta(\sigma_{13} + b\sigma_{23}) = C \quad (\tau_{12} + \beta\sigma_{12} \leqslant \tau_{23} + \beta\sigma_{23}), \tag{1b}$$

where τ_{13}, τ_{12} and τ_{23} are the principal shear stresses defined as $\tau_{13} = \frac{\sigma_1 - \sigma_3}{2}$, $\tau_{12} = \frac{\sigma_1 - \sigma_2}{2}$, $\tau_{23} = \frac{\sigma_2 - \sigma_3}{2}$ and σ_{13}, σ_{12} and σ_{23} are normal stresses acting on the planes of τ_{13}, τ_{12} and τ_{23} defined as $\sigma_{13} = \frac{\sigma_1 + \sigma_3}{2}$, $\sigma_{12} = \frac{\sigma_1 + \sigma_2}{2}$, $\sigma_{23} = \frac{\sigma_2 + \sigma_3}{2}$, in which σ_1, σ_2 and σ_3 are the principal stresses and satisfy $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. β and C are material parameters. The coefficient b is also a material parameter that reflects the influence of the intermediate principal shear stress as well as the intermediate principal stress, and $0 \leqslant b \leqslant 1$. The values of β, C and b can be evaluated by material tests. In practice, the USC is usually expressed in terms of the principal stresses as follows (Yu, 2004)

$$\sigma_1 - \frac{1 - \sin\varphi}{(1+b)(1+\sin\varphi)}(b\sigma_2 + \sigma_3) = \frac{2c\cos\varphi}{1+\sin\varphi} \quad \left(\sigma_2 \leqslant \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sin\varphi\right), \tag{2a}$$

$$\frac{1}{1+b}(\sigma_1 + b\sigma_2) - \frac{1 - \sin\varphi}{1+\sin\varphi}\sigma_3 = \frac{2c\cos\varphi}{1+\sin\varphi} \quad \left(\sigma_2 \geqslant \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sin\varphi\right), \tag{2b}$$

where c and φ are the cohesive strength and friction angle of the material, respectively.

Figure 1 shows the limiting loci of the USC in the deviatoric plane. Obviously, the USC is not a single strength criterion, but a series of piecewise linear strength criteria. The exact form of expression depends on the value of parameter b. When b varies from 0 to 1, a family of convex strength criteria including Mohr-Coulomb criterion (lower bound, 1900), the Generalized twin-shear strength criterion (upper bound, Yu et al., 1985) and a series of new in-between strength criteria are deduced, which in turn can suit for various different materials.

The parameter b plays an important role in the USC. It not only reflects the intermediate principal stress effect, but also determines the formulation of a strength criterion. So, it is called the strength criterion coefficient. The value of b can be determined by the true triaxial experiment of material. In practice, from the results of the true triaxial experiment, the limiting loci of the material in the deviatoric plane can be obtained. By comparing these loci to the deviatoric limiting loci of the USC,

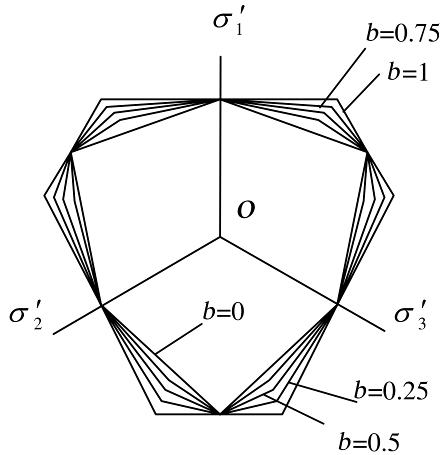

Fig. 1. The limiting loci of the USC in deviatoric plane

we will find one strength criterion of the USC to fit the experimental results well. This strength criterion and the corresponding value of b are for this sort of material. Then the strength criterion, i.e. the parameter b, is determined and the application is possible (see Yu, 2004).

In the latter sections of this paper, for convenience, we take compressive stress as positive. In that case the USC (2) is rewritten in the following form

$$\frac{1-\sin\varphi}{1+\sin\varphi}\sigma_1-\frac{1}{1+b}(b\sigma_2+\sigma_3)=\frac{2c\cos\varphi}{1+\sin\varphi}\quad\left(\sigma_2\leqslant\frac{\sigma_1+\sigma_3}{2}-\frac{\sigma_1-\sigma_3}{2}\sin\varphi\right),\tag{3a}$$

$$\frac{1-\sin\varphi}{(1+b)(1+\sin\varphi)}(\sigma_1+b\sigma_2)-\sigma_3=\frac{2c\cos\varphi}{1+\sin\varphi}\quad\left(\sigma_2\geqslant\frac{\sigma_1+\sigma_3}{2}-\frac{\sigma_1-\sigma_3}{2}\sin\varphi\right).\tag{3b}$$

3. Theoretical Formulation and Solutions

Figure 2 shows the problem under consideration – a circular opening in rock mass with internal radius r_{i} and external radius r_{o} subjected to the internal pressure p_{i} and the external pressure p_{o}. For simplicity, we adopt the following assumptions:

(1) the opening is in a state of plane strain;
(2) the rock mass is homogeneous and isotropic;
(3) the intact rock mass is linearly elastic up to failure, and the failed rock mass is perfectly plastic.

This is an axisymmetric problem and we use the cylindrical coordinate system (r, θ, z), where the z axis coincides with the axis of the opening. If p_{i} is below a critical value $p_{\mathrm{i}}^{\mathrm{cr}}$, a plastic zone of radius r_{c} will develop around the opening, and the

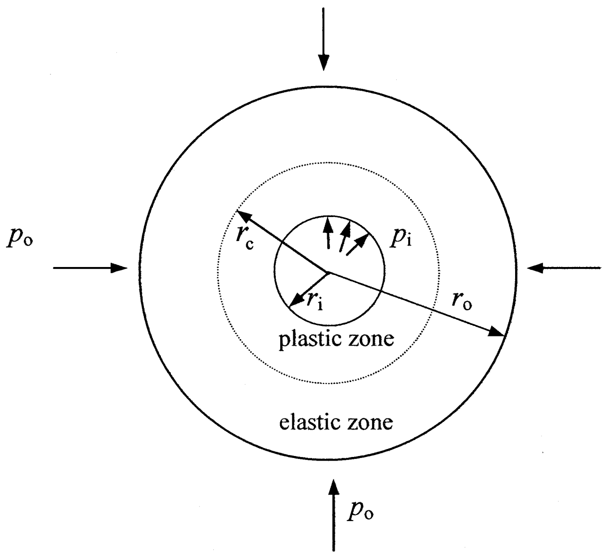

Fig. 2. Opening configuration and loads condition

rock mass outside the boundary defined by r_c remains elastic. In the following, we will derive the solutions for the stresses and displacement around the opening by incorporating the USC in detail.

3.1 Solution for Stresses and Extent of the Plastic Zone

For a plane strain body in the elasto-plastic state, the out-of-plane stress rapidly approaches the mean of the other two principal stresses throughout the plastic zone of the body as the in-plane plastic strains progressively increase. Hence, it is reasonable for us to adopt

$$\sigma_z = \frac{\sigma_r + \sigma_\theta}{2} \tag{4}$$

instead of the conventional condition under which σ_z is assumed to be intermediate between σ_r and σ_θ within the plastic zone of the opening, where σ_r, σ_θ and σ_z are radial, tangential and axial stresses, respectively (Bigoni and Laudiero, 1989). Then the USC for our case becomes

$$\frac{(2+b)(1-\sin\varphi)}{2+2b}\sigma_\theta - \frac{2+b+(2+3b)\sin\varphi}{2+2b}\sigma_r = 2c\cos\varphi, \tag{5}$$

which is obtained on substitution $\sigma_1 = \sigma_\theta$, $\sigma_2 = \sigma_z = (\sigma_r + \sigma_\theta)/2$ and $\sigma_3 = \sigma_r$ in Eq. (3b).

Combining Eqs. (4), (5), the equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{6}$$

and the boundary condition $\sigma_r = p_\mathrm{i}$ at $r = r_\mathrm{i}$, yields the stresses in the plastic zone as

$$\sigma_r = (p_\mathrm{i} + c\cot\varphi)(r/r_\mathrm{i})^{\frac{4(1+b)\sin\varphi}{(2+b)(1-\sin\varphi)}} - c\cot\varphi, \tag{7a}$$

$$\sigma_\theta = \frac{2+b+2\sin\varphi+3b\sin\varphi}{(2+b)(1-\sin\varphi)}(p_\mathrm{i} + c\cot\varphi)(r/r_\mathrm{i})^{\frac{4(1+b)\sin\varphi}{(2+b)(1-\sin\varphi)}} - c\cot\varphi, \tag{7b}$$

$$\sigma_z = \frac{2+b+b\sin\varphi}{(2+b)(1-\sin\varphi)}(p_\mathrm{i} + c\cot\varphi)(r/r_\mathrm{i})^{\frac{4(1+b)\sin\varphi}{(2+b)(1-\sin\varphi)}} - c\cot\varphi. \tag{7c}$$

The stresses in the elastic zone are expressed by Lame's solution as (see Timoshenko and Goodier, 1970)

$$\sigma_r = \frac{p_\mathrm{o}r_\mathrm{o}^2 - p_\mathrm{c}r_\mathrm{c}^2}{r_\mathrm{o}^2 - r_\mathrm{c}^2} - \frac{(p_\mathrm{o} - p_\mathrm{c})r_\mathrm{o}^2 r_\mathrm{c}^2}{r^2(r_\mathrm{o}^2 - r_\mathrm{c}^2)}, \tag{8a}$$

$$\sigma_\theta = \frac{p_\mathrm{o}r_\mathrm{o}^2 - p_\mathrm{c}r_\mathrm{c}^2}{r_\mathrm{o}^2 - r_\mathrm{c}^2} + \frac{(p_\mathrm{o} - p_\mathrm{c})r_\mathrm{o}^2 r_\mathrm{c}^2}{r^2(r_\mathrm{o}^2 - r_\mathrm{c}^2)}, \tag{8b}$$

$$\sigma_z = 2\nu\frac{p_\mathrm{o}r_\mathrm{o}^2 - p_\mathrm{c}r_\mathrm{c}^2}{r_\mathrm{o}^2 - r_\mathrm{c}^2}, \tag{8c}$$

where ν is Poisson's ratio of the rock mass, and p_c the radial stress at the elasto-plastic interface. Setting $r = r_\mathrm{c}$ in Eqs. (8a) and (8b), and substituting the results into Eq. (5), we get

$$p_\mathrm{c} = \frac{r_\mathrm{o}^2 p_\mathrm{o}(2+b)(1-\sin\varphi) - 2(r_\mathrm{o}^2 - r_\mathrm{c}^2)(1+b)c\cos\varphi}{r_\mathrm{o}^2(2+b+b\sin\varphi) - 2r_\mathrm{c}^2(1+b)\sin\varphi} \tag{9}$$

In view of the continuity of radial stress across the elasto-plastic interface, we have

$$(p_\mathrm{i} + c\cot\varphi)(r_\mathrm{c}/r_\mathrm{i})^{\frac{4(1+b)\sin\varphi}{(2+b)(1-\sin\varphi)}} - \frac{r_\mathrm{o}^2 p_\mathrm{o}(2+b)(1-\sin\varphi) - 2(r_\mathrm{o}^2 - r_\mathrm{c}^2)(1+b)c\cos\varphi}{r_\mathrm{o}^2(2+b+b\sin\varphi) - 2r_\mathrm{c}^2(1+b)\sin\varphi} - c\cot\varphi = 0, \tag{10}$$

from which the extent of the plastic zone r_c can be determined. On letting $r_\mathrm{c} = r_\mathrm{i}$ in above equation, the critical value of the internal pressure, p_i^cr, is obtained:

$$p_\mathrm{i}^\mathrm{cr} = \frac{r_\mathrm{o}^2 p_\mathrm{o}(2+b)(1-\sin\varphi) - 2(r_\mathrm{o}^2 - r_\mathrm{i}^2)(1+b)c\cos\varphi}{r_\mathrm{o}^2(2+b+b\sin\varphi) - 2r_\mathrm{i}^2(1+b)\sin\varphi}. \tag{11}$$

When $r_\mathrm{c} = r_\mathrm{o}$, the hollow cylinder is completely plastic. In that case the internal pressure p_i and the external pressure p_o satisfy the condition as

$$(p_\mathrm{i} + c\cot\varphi)(r_\mathrm{o}/r_\mathrm{i})^{\frac{4(1+b)\sin\varphi}{(2+b)(1-\sin\varphi)}} - p_\mathrm{o} - c\cot\varphi = 0, \tag{12}$$

which is obtained by setting $r_\mathrm{c} = r_\mathrm{o}$ in Eq. (10).

Once p_c and r_c are determined from Eqs. (9) and (10), the stresses in the elastic zone can be readily determined from Eq. (8).

We now see that due to the introduction of Eq. (4), the above derivations are simplified, which, however, leads to a discontinuity of the axial stress σ_z across the elasto-plastic interface. Evidently, this discontinuity will be eliminated if the material is assumed elastically incompressible ($\nu = 0.5$).

3.2 Solution for the Displacement

With the stresses around the opening, we now calculate the displacement in the elastic and plastic zones around the opening.

Using Lame's solution yields the displacement u in the elastic zone as

$$u = \frac{1}{2\mathrm{G}(r_o^2 - r_c^2)}\left[\frac{r_o^2 r_c^2 (p_o - p_c)}{r} + (1 - 2\nu)(r_o^2 p_o - r_c^2 p_c) r\right], \tag{13}$$

where G is the shear modulus of rock mass.

Hence, the displacement u_c at the elasto-plastic interface, $r = r_c$, is

$$u_c = \frac{r_c}{2G(r_o^2 - r_c^2)}[r_o^2(p_o - p_c) + (1 - 2\nu)(r_o^2 p_o - r_c^2 p_c)]. \tag{14}$$

In general, the displacement solution for the plastic zone of an elasto-plastic body is complicated, so some simplifying assumptions are introduced. The common practice is by using the flow rule for the plastic deformation zone. However, a number of tedious derivations are involved in obtaining the displacement solution by virtue of flow rule. Here we adopt the assumption of the material incompressibility within the plastic zone, which will result in a drastic simplification. In fact, this assumption means that the elastic portion of the strain becomes negligible comparing to the plastic portion of the strain within the plastic zone, and the plastic strain is governed by the non-associated flow rule with a zero dilation angle. If the nonzero strains in the plastic zone are denoted by ε_r and ε_θ, then we have

$$\varepsilon_r + \varepsilon_\theta = 0. \tag{15}$$

In view of the strain-displacement relations $\varepsilon_r = du/dr$ and $\varepsilon_\theta = u/r$, the above equation can be turned into

$$\frac{du}{dr} + \frac{u}{r} = 0. \tag{16}$$

Integrating the above equation, and using Eq. (14), we obtain displacement u in the plastic zone as

$$u = \frac{r_c^2}{2G(r_o^2 - r_c^2) r}[r_o^2(p_o - p_c) + (1 - 2\nu)(r_o^2 p_o - r_c^2 p_c)]. \tag{17}$$

At the internal wall of the opening, $r = r_i$, the displacement u_i is

$$u_i = \frac{r_c^2}{2G(r_o^2 - r_c^2) r_i}[r_o^2(p_o - p_c) + (1 - 2\nu)(r_o^2 p_o - r_c^2 p_c)]. \tag{18}$$

This equation, together with Eqs. (9) and (10), gives the relationship between the displacement u_i on the internal wall of the opening and the internal pressure p_i, and is called the ground response equation of the opening.

Clearly, the stresses, displacement, the extent of the plastic zone and the ground response equation of an opening all depend on the parameter *b*. Hence, the present solutions have capability to take account of the intermediate principal stress effect.

3.3 Solution for the Case of a Tunnel Opening

In the special case of the opening with the external radius $r_o = \infty$, the problem considered above becomes an opening in an infinite medium, which is a model of a deep circular tunnel subjected to the support pressure p_i and the initial stress p_o.

Taking $r_o = \infty$ in Eqs. (10), (11) and (9), respectively, and after mathematical manipulations, we have the extent of the plastic zone of the tunnel as

$$r_c = r_i \left[\frac{(2+b)(1-\sin\varphi)(p_o + c\cot\varphi)}{(2+b+b\sin\varphi)(p_i + c\cot\varphi)} \right]^{\frac{(2+b)(1-\sin\varphi)}{4(1+b)\sin\varphi}}, \tag{19}$$

and the critical value of the support pressure and the radial stress at the elasto-plastic interface as

$$p_i^{cr} = p_c = \frac{p_o(2+b)(1-\sin\varphi) - 2(1+b)c\cos\varphi}{2+b+b\sin\varphi}. \tag{20}$$

For the stresses in the plastic zone, Eq. (7) is still valid.

The true displacement of the tunnel (still denoted by u) is the one of the rock mass induced by excavation, and approaches to zero at infinity, which is, obviously, not the displacement given in Section 3.2 as $r_o = \infty$. When $r_o = \infty$, subtracting the displacement (i.e. $(1-2\nu)p_o r/2G$, Reed, 1986) occurring in rock mass under initial stress p_o from the displacement given by Eq. (13), we can obtain the displacement u in the elastic zone of the tunnel. The result is

$$u = \frac{r_c^2}{2Gr}(p_o - p_c). \tag{21}$$

Then, through operations similar to those in Section 3.2, and using Eqs. (19) and (20), we can obtain the displacement u in the plastic zone as

$$u = \frac{(1+b)(c\cos\varphi + p_o\sin\varphi)r_i^2}{(2+b+b\sin\varphi)Gr} \left[\frac{(2+b)(1-\sin\varphi)(p_o + c\cot\varphi)}{(2+b+b\sin\varphi)(p_i + c\cot\varphi)} \right]^{\frac{(2+b)(1-\sin\varphi)}{2(1+b)\sin\varphi}}, \tag{22}$$

and the wall displacement u_i of the tunnel as

$$u_i = \frac{(1+b)(c\cos\varphi + p_o\sin\varphi)r_i}{(2+b+b\sin\varphi)G} \left[\frac{(2+b)(1-\sin\varphi)(p_o + c\cot\varphi)}{(2+b+b\sin\varphi)(p_i + c\cot\varphi)} \right]^{\frac{(2+b)(1-\sin\varphi)}{2(1+b)\sin\varphi}}. \tag{23}$$

Similar to Eq. (18), Eq. (23) is called the ground response equation of the tunnel. It forms the basis for tunnel support design in the convergence-confinement method.

4. Application of the Present Solutions

In this section, we will derive some specific solutions and investigate the influence of the intermediate principal stress on the stress distributions, the extent of the plastic zone, the wall displacement and the ground response curve of a tunnel by applying the present solutions.

4.1 Fenner's Formula and Kastner's Formula

In Section 3, we derived the solutions for the opening problem involving the parameter *b*. Here we discuss the solutions for the special case of $b = 0$. As an example, we consider the opening in an infinite medium, which corresponds to a deep tunnel without the intermediate principal stress effect. For this case, the solutions can be obtained by setting $b = 0$ in the relevant expressions in Section 3. The primary results are listed below:

Solution for the stresses in the plastic zone

$$\sigma_r = (p_\mathrm{i} + c\cot\varphi)(r/r_\mathrm{i})^{\frac{2\sin\varphi}{1-\sin\varphi}} - c\cot\varphi, \tag{24a}$$

$$\sigma_\theta = \frac{1+\sin\varphi}{1-\sin\varphi}(p_\mathrm{i} + c\cot\varphi)(r/r_\mathrm{i})^{\frac{2\sin\varphi}{1-\sin\varphi}} - c\cot\varphi, \tag{24b}$$

$$\sigma_z = \frac{1}{1-\sin\varphi}(p_\mathrm{i} + c\cot\varphi)(r/r_\mathrm{i})^{\frac{2\sin\varphi}{1-\sin\varphi}} - c\cot\varphi; \tag{24c}$$

Solution for the extent of the plastic zone

$$r_\mathrm{c} = r_\mathrm{i}\left[\frac{(1-\sin\varphi)(p_\mathrm{o} + c\cot\varphi)}{(p_\mathrm{i} + c\cot\varphi)}\right]^{\frac{1-\sin\varphi}{2\sin\varphi}}; \tag{25}$$

Solution for the critical internal pressure and the radial stress at the elasto-plastic interface

$$p_\mathrm{i}^\mathrm{cr} = p_\mathrm{c} = p_\mathrm{o}(1-\sin\varphi) - c\cos\varphi; \tag{26}$$

Solution for the wall displacement

$$u_\mathrm{i} = \frac{(c\cos\varphi + p_\mathrm{o}\sin\varphi)r_\mathrm{i}}{2G}\left[\frac{(1-\sin\varphi)(p_\mathrm{o} + c\cot\varphi)}{p_\mathrm{i} + c\cot\varphi}\right]^{\frac{1-\sin\varphi}{\sin\varphi}}. \tag{27}$$

In fact, Eq. (25) will become Kastner's formula as follows (Kastner, 1962)

$$r_\mathrm{c} = r_\mathrm{i}\left[\frac{2}{m+1}\cdot\frac{\sigma_\mathrm{c} + p_\mathrm{o}(m-1)}{\sigma_\mathrm{c} + p_\mathrm{i}(m-1)}\right]^{\frac{1}{m-1}}, \tag{28}$$

provided that the form is re-expressed in terms of m and σ_c instead of c and φ. m and σ_c are compressive-tensile strength ratio and uniaxial compressive strength of material, and relate to c and φ by the following:

$$m = \frac{1+\sin\varphi}{1-\sin\varphi}, \qquad \sigma_\mathrm{c} = \frac{2c\cos\varphi}{1-\sin\varphi}. \tag{29a, b}$$

Furthermore, if the cohesion of the material at the elasto-plastic interface is neglected, p_c is then given by Eq. (26) with $c = 0$. The corresponding values of r_c and u_i are

$$r_\mathrm{c} = r_\mathrm{i}\left[\frac{p_\mathrm{o}(1-\sin\varphi) + c\cot\varphi}{p_\mathrm{i} + c\cot\varphi}\right]^{\frac{1-\sin\varphi}{2\sin\varphi}}, \tag{30}$$

$$u_\mathrm{i} = \frac{(c\cos\varphi + p_\mathrm{o}\sin\varphi)r_\mathrm{i}}{2G}\left[\frac{p_\mathrm{o}(1-\sin\varphi) + c\cot\varphi)}{p_\mathrm{i} + c\cot\varphi}\right]^{\frac{1-\sin\varphi}{\sin\varphi}}. \tag{31}$$

Clearly, Eq. (30) is equivalently identical to Fenner's formula (see Talobre, 1957).

Now we see that Fenner's formula and Kastner's formula are in effect special cases of the present solution.

4.2 An Example

To investigate the effect of the intermediate principal stress on the ground response curve, the stress distributions, the extent of the plastic zone and the wall displacement, a deep circular tunnel is analysed. Data for the tunnel are as follows: shear modulus $G = 5 \times 10^3\, MPa$, cohesion $c = 2.9\, MPa$, friction angle $\varphi = 30°$, initial stress $p_o = 40\, MPa$, radius of the tunnel $r_i = 3m$.

The ground response curve of the tunnel for different values of b is given in Fig. 3. The abscissa represents the dimensionless wall displacement, u_i/r_i, and the ordinate

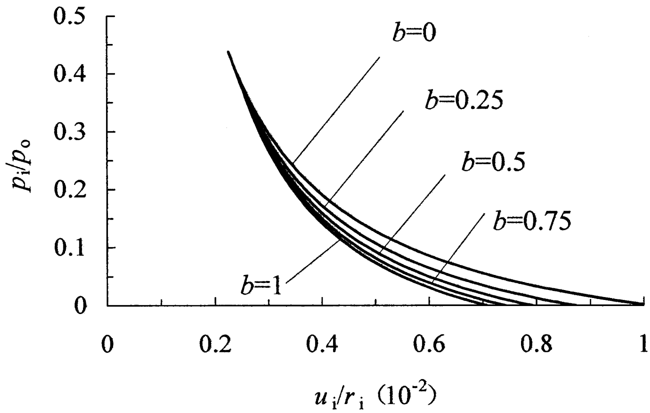

Fig. 3. Ground response curves for different values of b

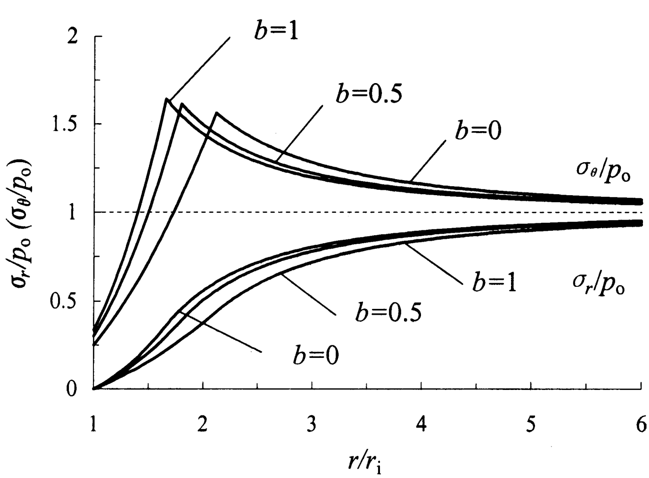

Fig. 4. Stress distribution around the tunnel for different value of b

Table 1. Dimensionless values of the extent of the plastic zone and wall displacement for different values of b

b	0	0.25	0.5	0.75	1
r_c/r_i	2.17	1.91	1.80	1.71	1.66
$u_i/r_i(10^{-2})$	1.01	0.87	0.79	0.74	0.71

the dimensionless support pressure, p_i/p_o. The curve for $b = 0$ is equivalent to the result based on Mohr-Coulomb criterion. From Fig. 3, we see that the changing of b will result in the variation of the ground response curve.

Figure 4 illustrates the radial and tangential stress distributions along the radial direction of the tunnel for zero support pressure ($p_i = 0$) and a series of values of b. It is seen that the tangential stress reaches its maximum value at the elasto-plastic interface, and the maximum value increases and the radius of the plastic zone decreases with b, while the radial stress increases with the radius monotonically, and decreases with b.

Table 1 lists values of r_c/r_i and values of u_i/r_i for $p_i = 0$ and a series of values of b. r_c/r_i is equal to 2.23 from Fenner's formula, and u_i/r_i is equal to 0.0112 from Eq. (31). It is seen that Fenner's formula and Kastner's formula (in case of $b = 0$ in present solution) give the broadest and the second broadest extent of the plastic zone, respectively. The extent of the plastic zone from present solution with $b \neq 0$ is smaller than that from Kastner's formula and decreases with b. It is also seen that the variation relation of the wall displacement u_i with b is similar to that of the extent of the plastic zone r_c with b. Therefore, the extent of the plastic zone and the wall displacement will become smaller when the intermediate principal stress effect is taken into account.

5. Conclusions

Analytical solutions are presented for stress distributions, displacement distribution and the extent of the plastic zone of a circular opening with a finite external radius subjected to uniform pressures acting on the internal and external boundary surfaces under plane strain conditions. The solutions for a circular opening in an infinite medium as a special case are also given. In deriving these solutions, the unified strength criterion due to Yu and He (1991) is used to simulate rock mass behavior. As a result, the present solutions involve the parameter b originally appearing in the unified strength criterion, and can take account of the intermediate principal stress effect of rock mass quantitatively and may be applied to a wide range of rock mass. By investigating them, we see that Fenner's formula and Kastner's formula are all special cases of the present solutions.

Finally, a deep circular tunnel is studied. The results show that the intermediate principal stress has an influence on the ground response curve, the stress distributions, the extent of the plastic zone and the wall displacement of the tunnel, and Fenner's formula and Kastner's formula give the largest and the second largest extent of the plastic zone. This means that Fenner's formula and Kastner's formula may underestimate rock mass strength. Hence, the intermediate principal stress effect of rock mass should be taken into account in order to get an accurate prediction of response of a tunnel.

The present solutions are limited to a circular opening in case of axisymmetric deformation in perfectly plastic rock mass. For more complex problems, it will be difficult to achieve such analytical solutions by using the unified strength criterion. However, the unified strength criterion can be readily used for complex problems in conjunction with numerical methods.

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