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Two Ordinary Kriging Approaches to Predicting Block Grade Distributions[1]

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Multigaussian kriging aims at estimating the local distributions of regionalized variables and functions of these variables (transfer or recovery functions) at unsampled locations. In this paper, we focus on the evaluation of the recoverable reserves in an ore deposit accounting for a change of support and information effect caused by ore/waste misclassifications. Two approaches are proposed: the multigaussian model with Monte Carlo integration and the discrete Gaussian model. The latter is simpler to use but requires stronger hypotheses than the former. In each model, ordinary multigaussian kriging gives unbiased estimates of the recoverable reserves that do not utilize the mean value of the normal score data.

The concepts are illustrated through a case study on a copper deposit which shows that local estimates of the metal content based on ordinary multigaussian kriging are close to the optimal conditional expectation when the data are abundant and are not dominated by the global mean when the data are scarce. The two proposed approaches (Monte Carlo integration and discrete Gaussian model) lead to similar results when compared to two other geostatistical methods: service variables and ordinary indicator kriging, which show strong deviations from conditional expectation.

KEY WORDS: change of support, information effect, multigaussian kriging, discrete Gaussian model, Monte Carlo integration, conditional expectation

INTRODUCTION

Geostatistical studies often aim at quantifying the uncertainty in estimation of unsampled values of regionalized variables, which plays an important role in risk assessment and decision-making. In many fields of application, e.g. ore reserve evaluation, reservoir characterization, hydrogeology, soil and environmental sciences, the quantities of interest (grade, permeability, conductivity, nutrient or contaminant concentration, etc.) refer to bigger supports than that of the available samples. To account for such a change of support in geostatistical estimations, several approaches have been developed, including indicator kriging with an affine or a lognormal correction, disjunctive kriging and conditional expectation (Journel,

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1984; Rivoirard, 1994; Chilès and Delfiner, 1999, p. 435). The first two methods only make use of the two-point distributions of the available data. In contrast, the latter takes advantage of their multivariate distribution. Both disjunctive kriging and conditional expectation rely on the mean value of (a transform of) the variable of interest, which may be considered demanding and undesirable. To avoid using this mean value, alternative techniques have been proposed, such as service variables or uniform conditioning (Guibal and Remacre, 1984; Remacre, 1989; Rivoirard, 1994, p. 96), but the theoretical bases of these are not always sound. Another option is the so-called ordinary multigaussian kriging (Emery, 2005b), which relies on the multivariate distribution of the data but avoids the use of their mean value.

This paper focuses on the ordinary multigaussian kriging approach to the local estimation of recovery functions with a change of support. It has three main objectives: (1) to provide a simple demonstration of the unbiasedness of ordinary multigaussian kriging, (2) to propose two different ways for dealing with change of support in recovery estimation problems, and (3) to generalize the ordinary multigaussian kriging approach to the modeling of the information effect that arises in resource evaluation problems. Although the terminology refers to mining applications, the concepts and results presented hereafter are of interest to other fields of application confronted with problems of change of support and information effect, e.g. forest inventories, agricultural land management, remediation of contaminated areas, upscaling problem in hydrogeology and petroleum engineering.

AIMS AND SCOPE

Support Effect

Let $\{Z_{\mathbf{x}}, \mathbf{x} \in \mathsf{R}^d\}$ be the random field that represents the regionalized variable under study, say the grade of an element of interest in an ore deposit. Henceforth, the analysis focuses on a particular block of the deposit, denoted by v. The regularized variable over v is defined as the arithmetic average of the point-support values within v:

$$Z_v = \frac{1}{|v|} \int_v Z_{\mathbf{x}} \, \mathrm{d}\mathbf{x} \tag{1}$$

where $|v|$ stands for the volume of block v.

It is of interest to estimate functions of Z_v (these are called "transfer functions" or, in the mining jargon, "recovery functions" as they are used to quantify the recoverable reserves in the deposit). For instance, one looks for the probability that Z_v falls short of a given economic cutoff z, or for the expected metal quantity above such a cutoff:

$$Q(z) = Z_v \mathrm{I}(Z_v; z) \tag{2}$$

where $\mathrm{I}(Z_v; z)$ is an indicator function, equal to 1 if $Z_v > z$, 0 otherwise. When varying the cutoff, one sees that the problem amounts to estimating the distribution of Z_v conditional to the available point-support data.

Information Effect

In resource-reserve assessment problems, the classification of Z_v as above or below a given cutoff is made on incomplete information. For instance, in open pit mining, the blocks are sent to mill or to dump according to their grades estimated from blast hole data, not their true grades, hence they may be misclassified. This "information effect" implies a degradation of the value of the ore mined out with respect to the ideal case of a perfect ore-waste classification.

Although the classification is made on the basis of an estimate of the block grade, say Z_v^*, the recovered grade is the true one (Z_v). Hence, the recovered metal quantity at cutoff z is:

$$Q^*(z) = Z_v \mathrm{I}(Z_v^*; z) \tag{3}$$

This metal quantity is traditionally referred to as an *indirect* or *actual* recovery function, as opposed to the *direct* or *ideal* recovery (Eq. (2)) (Matheron, 1976; Emery and Soto Torres, 2005). One may also define the recovery associated with the grade estimate:

$$Q^{**}(z) = Z_v^* \mathrm{I}(Z_v^*; z) \tag{4}$$

which is expected to match the indirect recovery (Eq. (3)) if Z_v^* is conditionally unbiased (Emery and Soto Torres, 2005, p. 53).

The general problem is to estimate functions of the type $\varphi(Z_v, Z_v^*)$. In practice, the data from which Z_v^* is constructed (blast holes) are not available and one wishes to calculate the direct and indirect recovery functions from exploration data (drill holes). This problem requires knowing the locations $\{\mathbf{u}'_1, \ldots \mathbf{u}'_K\}$ of the blast holes that will be used to ultimately estimate the block grade, typically by a weighted average of the form:

$$Z_v^* = \sum_{k=1}^{K} \lambda_k Z_{\mathbf{u}'_k} \tag{5}$$

If the pattern of blast hole sampling is available, the weights $\{\lambda_k, k = 1 \ldots K\}$ can be determined without knowing the blast hole grades, as they usually depend only on the sampling configuration (e.g. kriging or inverse distance weighting).

Hypotheses

To deal with support and information effects, two approaches will be proposed: the first one is based on a discretization of v by several points and a Monte Carlo integration; the second one relies on the so-called discrete Gaussian model. Both approaches work with Gaussian variables and require performing a normal score transform of the initial variable:

$$\forall \mathbf{x} \in \mathsf{R}^d, Z_{\mathbf{x}} = \phi(Y_{\mathbf{x}}) \tag{6}$$

where ϕ is the point-support transformation function. Hereafter, the following assumptions are made:

1. the transformed variable $\{Y_{\mathbf{x}}, \mathbf{x} \in \mathsf{R}^d\}$ has a multivariate Gaussian distribution with mean m and unit variance;
2. the available data are the point-support values at locations $\{\mathbf{x}_\alpha, \alpha = 1 \ldots n\}$ (exploration data).

In practice, the data are transformed to standard Gaussian values, i.e. the mean m is set to zero and the transformation function ϕ (Eq. (6)) is determined accordingly (Rivoirard, 1994, p. 46). However, in this paper, we will leave the generic value m and seek estimators of recovery functions that do not depend on m. This way, the proposed estimators will not be dominated by the global mean of normal scores when the data are scarce in the estimation neighborhood, and hence remain unbiased even if this mean is misspecified or is unsuitable at a local scale.

FIRST APPROACH: MONTE CARLO INTEGRATION

The principle of Monte Carlo integration is to approximate the block value by the average of M points $\{\mathbf{u}_1, \ldots \mathbf{u}_M\}$ that discretize this block:

$$Z_v \approx \frac{1}{M} \sum_{i=1}^{M} Z_{\mathbf{u}_i} = \frac{1}{M} \sum_{i=1}^{M} \phi(Y_{\mathbf{u}_i}) \tag{7}$$

Any function of Z_v is therefore a function of a Gaussian random vector $\boldsymbol{Y} = (Y_{\mathbf{u}_1}, \ldots Y_{\mathbf{u}_M})$ with M correlated components. Thereby, the change-of-support problem amounts to estimating a multivariate function, say $\varphi(\boldsymbol{Y})$. The components of $\boldsymbol{Y}$ can also include the blast hole values $(Y_{\mathbf{u}'_1}, \dots Y_{\mathbf{u}'_K})$ so as to deal with the information effect (Eq. (5)). For instance, the indirect metal quantity at cutoff z (Eq. (3)) corresponds to the following transfer function:

$$\varphi(Y_{\mathbf{u}_1}, \dots Y_{\mathbf{u}_M}; Y_{\mathbf{u}'_1}, \dots Y_{\mathbf{u}'_K}) \approx \left(\frac{1}{M} \sum_{i=1}^{M} \phi(Y_{\mathbf{u}_i}) \right) \times I \left(\sum_{k=1}^{K} \lambda_k \phi(Y_{\mathbf{u}'_k}); z \right) \tag{8}$$

Conditional Expectation of a Gaussian Random Vector

Under the multivariate Gaussian assumption, the distribution of $\boldsymbol{Y}$ conditional to the data at $\{\mathbf{x}_\alpha, \alpha = 1 \dots n\}$ is Gaussian and is characterized by its first and second order moments. The conditional mean of $\boldsymbol{Y}$ is its simple kriging $\boldsymbol{Y}^{\mathrm{SK}}$, whereas the conditional variance-covariance matrix of $\boldsymbol{Y}$ is the variance-covariance matrix of the simple kriging errors $\boldsymbol{\Sigma}^{\mathrm{SK}}$. This matrix is symmetric and positive semi-definite, hence it can be written as follows:

$$\boldsymbol{\Sigma}^{\mathrm{SK}} = (\sqrt{\boldsymbol{\Sigma}^{\mathrm{SK}}})^2 \tag{9}$$

where $\sqrt{\boldsymbol{\Sigma}^{\mathrm{SK}}}$ is a symmetric, positive semi-definite matrix called *square root* of $\boldsymbol{\Sigma}^{\mathrm{SK}}$. Such decomposition is unique and is found by diagonalizing $\boldsymbol{\Sigma}^{\mathrm{SK}}$ in an orthonormal basis and taking the square root of the entries of the diagonal matrix. Accordingly, one has:

$$\boldsymbol{Y} = \boldsymbol{Y}^{\mathrm{SK}} + \sqrt{\boldsymbol{\Sigma}^{\mathrm{SK}}}\boldsymbol{T} \tag{10}$$

where $\boldsymbol{T}$ is a standard Gaussian vector with components uncorrelated among themselves and independent of the data.

By definition, the conditional expectation of $\varphi(\boldsymbol{Y})$ is the expected value of the conditional distribution of $\varphi(\boldsymbol{Y})$:

$$[\varphi(\boldsymbol{Y})]^{\mathrm{CE}} = \mathrm{E}\{\varphi(\boldsymbol{Y})|\mathrm{data}\} = \int \varphi(\boldsymbol{Y}^{\mathrm{SK}} + \sqrt{\boldsymbol{\Sigma}^{\mathrm{SK}}}\boldsymbol{t})g(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \tag{11}$$

where $g(\boldsymbol{t})$ is the *pdf* of $\boldsymbol{T}$. In practice, the integral is calculated numerically by drawing a large number of independent realizations of $\boldsymbol{T}$, say $\{\boldsymbol{t}_1, \dots \boldsymbol{t}_N\}$, and by averaging the results obtained in each realization:

$$[\varphi(\boldsymbol{Y})]^{\mathrm{CE}} \approx \frac{1}{N}\sum_{i=1}^{N}\varphi\left(\boldsymbol{Y}^{\mathrm{SK}} + \sqrt{\boldsymbol{\Sigma}^{\mathrm{SK}}}\boldsymbol{t}_i\right) \tag{12}$$

This approach for calculating the conditional expectation with a change of support has been proposed and applied by Verly (1984, 1986).

Construction of an Unbiased Estimator

In this subsection, we look for a generalization of the conditional expectation that omits the mean of the random field $\{Y_{\mathbf{x}}, \mathbf{x} \in \mathsf{R}^d\}$. In Equation (11), $\boldsymbol{Y}^{\mathrm{SK}}$ is a Gaussian vector with mean $\boldsymbol{m} = (m, \ldots m)$ and covariance-variance matrix $\mathbf{S}^{\mathrm{SK}}$ such that:

$$\mathbf{S}^{\mathrm{SK}} = \mathbf{R} - \boldsymbol{\Sigma}^{\mathrm{SK}} \tag{13}$$

where $\mathbf{R}$ is the prior correlation matrix of $\boldsymbol{Y}$. Note that $\mathbf{S}^{\mathrm{SK}}$ is independent of $\boldsymbol{Y}^{\mathrm{SK}}$ since the kriging error covariances are data-independent. The conditional expectation estimator (Eq. (11)) therefore belongs to the family of estimators of the form:

$$[\varphi(\boldsymbol{Y})]^* = \int \varphi(\boldsymbol{V} + \sqrt{\mathbf{R} - \mathbf{S}}\boldsymbol{t})g(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \quad \text{with} \quad \boldsymbol{V} \sim N(\boldsymbol{m}, \mathbf{S}) \tag{14}$$

for any covariance-variance matrix $\mathbf{S}$ such that $\mathbf{R} - \mathbf{S}$ is positive semi-definite. A key result is that $[\varphi(\boldsymbol{Y})]^*$ estimates $\varphi(\boldsymbol{Y})$ without bias: indeed, if $\boldsymbol{V}$ and $\mathbf{S}$ fulfill the previous conditions and if $\boldsymbol{T}$ is an independent standard Gaussian vector with components uncorrelated among themselves, then

$$E\left\{\int \varphi(\boldsymbol{V} + \sqrt{\mathbf{R} - \mathbf{S}}\,\boldsymbol{t})g(\boldsymbol{t})\mathrm{d}\boldsymbol{t}\right\} = E[\varphi(\boldsymbol{V} + \sqrt{\mathbf{R} - \mathbf{S}}\,\boldsymbol{T})] = E[\varphi(\boldsymbol{Y})] \tag{15}$$

The last equality stems from the fact that $\boldsymbol{Y}$ and $\boldsymbol{V} + \sqrt{\mathbf{R} - \mathbf{S}}\boldsymbol{T}$ have a Gaussian distribution with mean $\boldsymbol{m}$ and covariance matrix $\mathbf{R}$. *Ordinary multigaussian kriging* is defined by replacing $\boldsymbol{V}$ by the ordinary kriging of $\boldsymbol{Y}$ ($\boldsymbol{Y}^{\mathrm{OK}}$) and $\mathbf{S}$ by the covariance-variance matrix of the ordinary kriging estimator ($\mathbf{S}^{\mathrm{OK}}$), giving:

$$[\varphi(\boldsymbol{Y})]^{\mathrm{oMK}} = \int \varphi(\boldsymbol{Y}^{\mathrm{OK}} + \sqrt{\mathbf{R} - \mathbf{S}^{\mathrm{OK}}}\boldsymbol{t})g(\boldsymbol{t})\mathrm{d}\boldsymbol{t} \tag{16}$$

The condition that $\mathbf{R}$–$\mathbf{S}^{\mathrm{OK}}$ is positive semi-definite has to be checked, e.g. by computing the eigenvalues of the matrix. It is not automatically fulfilled: for instance, in case of a point-support estimation ($M = 1$), one obtains a negative term under the square root in Equation (16) if the estimator has a variance greater than one, which may occur if negative kriging weights are present (Emery, 2006a). Another example is the case of a pure nugget effect, for which $\mathbf{R}$–$\mathbf{S}^{\mathrm{OK}}$ is positive semi-definite only if the number of data is greater than or equal to the number of points discretizing v ($n \geq M$). Like for the classical conditional expectation, the integral in Equation (16) is calculated numerically by putting:

$$[\varphi(\boldsymbol{Y})]^{\mathrm{oMK}} \approx \frac{1}{N} \sum_{i=1}^{N} \varphi\left(\boldsymbol{Y}^{\mathrm{OK}} + \sqrt{\mathbf{R} - \mathbf{S}^{\mathrm{OK}}}\,\boldsymbol{t}_i\right) \tag{17}$$

where $\{\boldsymbol{t}_1, \ldots \boldsymbol{t}_N\}$ are independent realizations of $\boldsymbol{T}$.

Outline of the Method and Comments

The following algorithm summarizes the previous statements and the steps required for using the Monte Carlo approach.

1. Find a representative distribution of the original data (a declustering technique may be needed at this stage).
2. Transform these data into standard Gaussian values ($m = 0$); obtain the transformation function ϕ (Eq. (6)).
3. Discretize v into M points and determine the transfer function $\varphi(\boldsymbol{Y})$ to be estimated.
4. Model the covariance of the normal score data and calculate the correlation matrix $\mathbf{R}$.
5. Calculate the ordinary kriging estimator $\boldsymbol{Y}^{\mathrm{OK}}$ and the associated covariance-variance matrix $\mathbf{S}^{\mathrm{OK}}$.
6. Simulate a large set of independent Gaussian vectors $\{\boldsymbol{t}_1, \ldots \boldsymbol{t}_N\}$ with M uncorrelated components and estimate $\varphi(\boldsymbol{Y})$ by applying Equation (17).

The Monte Carlo approach is theoretically sound (up to the discretization of v into M points) and does not require any assumption other than the ones described in the previous section. It allows one to account for support and information effects and can be extended to the estimation of non-additive variables, for which Equation (1) is no longer valid (e.g. a solubility ratio in mining applications, a permeability for aquifer or reservoir modeling, or a pH in agricultural land management): the definition of the transfer function $\varphi(\boldsymbol{Y})$ is extremely general and does not require that the regularized value Z_v is the arithmetic average of the point-support values within v. In contrast, the Monte Carlo approach is often CPU intensive and does not directly provide a measure of precision such as an estimation variance. A second possibility is to perform the change of support under the hypotheses of the discrete Gaussian model, which is detailed in the next section.

SECOND APPROACH: DISCRETE GAUSSIAN MODEL

Presentation of the Model

Here, space is viewed as the union of non-overlapping blocks which are identical up to a translation, and the point-support locations are uniformly randomized within the blocks they belong to (Rivoirard, 1994, p. 88). Henceforth, to avoid confusion with fixed locations, the random locations will be denoted by $\underline{\mathbf{x}}$ instead of $\mathbf{x}$. The point-support variable $\{Z_{\underline{\mathbf{x}}}, \underline{\mathbf{x}} \in \mathsf{R}^d\}$ is transformed into a Gaussian random field with mean m and unit variance: $\{Y_{\underline{\mathbf{x}}}, \underline{\mathbf{x}} \in \mathsf{R}^d\}$ (Eq. (6)). Similarly, the regularized variable Z_v (Eq. (1)) can be transformed into a Gaussian variable Y_v with mean m_v and unit variance:

$$Z_v = \phi_v(Y_v) \tag{18}$$

The basic hypothesis of the discrete Gaussian model is that the joint distribution of $\{Y_{\underline{\mathbf{x}}}, \underline{\mathbf{x}} \in \mathsf{R}^d\}$ and Y_v is multivariate Gaussian. Under this hypothesis, when $\underline{\mathbf{x}}$ is a randomly chosen point in v, the pair $\{Y_{\underline{\mathbf{x}}}, Y_v\}$ is bigaussian; let r be its correlation coefficient (*change-of-support coefficient*). Then:

$$\forall \underline{\mathbf{x}} \in v,\, Y_{\underline{\mathbf{x}}} - m = r(Y_v - m_v) + \sqrt{1 - r^2}T \tag{19}$$

where T is a standard Gaussian variable independent of Y_v. The model is constructed so as to honor Cartier's relation, which states that the expected value of a sample picked up randomly in a block with known value is equal to the block value (Matheron, 1984, p. 425; Chilès and Delfiner, 1999, p. 426):

$$\forall \underline{\mathbf{x}} \in v,\, Z_v = E(Z_{\underline{\mathbf{x}}} | Z_v) \tag{20}$$

Because of Equations (18) and (19), one has:

$$\phi_v(Y_v) = E(\phi(Y_{\underline{\mathbf{x}}})|Y_v) = \int \phi(rY_v + m - rm_v + \sqrt{1 - r^2}t)g(t)\,\mathrm{d}t \tag{21}$$

which relates the transformation functions at both supports (ϕ and ϕ_v), the change-of-support coefficient r and the means m and m_v. In practice, the latter are assumed to be zero, i.e. one works with standard Gaussian variables, so that ϕ_v is defined as follows (Rivoirard, 1994, p. 82):

$$\phi_v(Y_v) = \int \phi(rY_v + \sqrt{1 - r^2}t)g(t)\mathrm{d}t \tag{22}$$

By comparing Equations (21) and (22), the following relationship is found:

$$m = rm_v \tag{23}$$

This way, the model built upon Eq. (22) honors Cartier's relation and is consistent, even if m is not exactly zero, i.e. it is robust to a misspecification of the normal score mean value. In other words, if ϕ transforms $Z_{\underline{\mathbf{x}}}$ into a Gaussian variable $Y_{\underline{\mathbf{x}}}$ with mean m (instead of 0), then the function ϕ_v defined in Equation (22) transforms Z_v into a Gaussian variable Y_v with mean $m_v = m/r$.

Given a set of point-support data at locations $\{\underline{\mathbf{x}}_\alpha, \alpha = 1 \ldots n\}$, the conditional distribution of Y_v is still Gaussian, with mean equal to its simple kriging from the data (Y_v^{SK}) and variance equal to the simple kriging variance $(\sigma_v{}^{\mathrm{SK}})^2$ (Rivoirard, 1994, p. 94; Chilès and Delfiner, 1999, p. 445). The conditional expectation of a recovery function $\varphi(Y_v)$ is defined as:

$$[\varphi(Y_v)]^{\mathrm{CE}} = \mathrm{E}\{\varphi(Y_v)|\mathrm{data}\} = \int \varphi\left(Y_v^{\mathrm{SK}} + \sigma_v^{\mathrm{SK}}t\right) g(t)\, \mathrm{d}t \tag{24}$$

and explicitly depends on the means m and m_v, through the simple kriging term Y_v^{SK}.

Use of Ordinary Kriging

One is interested in an unbiased estimator of $\varphi(Y_v)$ that avoids use of the mean values m and m_v. First, note that the simple kriging estimator Y_v^{SK}. in Equation (24) is a Gaussian variable with mean m_v and variance

$$\left(s_v^{\mathrm{SK}}\right)^2 = 1 - \left(\sigma_v^{\mathrm{SK}}\right)^2 \tag{25}$$

The conditional expectation estimator is therefore of the form

$$[\varphi(Y_v)]^* = \int \varphi(V + \sqrt{1 - s^2}t)g(t)\mathrm{d}t \quad \text{with} \quad V \sim N(m_v, s^2) \tag{26}$$

which constitutes a family of unbiased estimators of $\varphi(Y_v)$, see Equation (15). For instance, the *ordinary multigaussian kriging* estimator is defined by using the ordinary kriging of Y_v instead of its simple kriging:

$$[\varphi(Y_v)]^{\mathrm{oMK}} = \int \varphi(Y_v^{\mathrm{OK}} + \sqrt{1 - \left(s_v^{\mathrm{OK}}\right)^2} t) g(t)\, \mathrm{d}t \tag{27}$$

where $(s_v^{\mathrm{OK}})^2$ is the variance of ${Y_v}^{\mathrm{OK}}$. Note that, because the mean of the point-support data is r times the mean of Y_v (Eq. (23)), the traditional ordinary kriging system has to be modified to ensure unbiasedness; in particular the kriging weights must add to $1/r$ (Emery, 2005b, p. 308). The estimator in Equation (27) no longer uses the mean value of the normal score data. It can be calculated via Monte Carlo integration, by drawing many independent standard Gaussian values $\{t_1, \ldots t_N\}$ and putting:

$$[\varphi(Y_v)]^{\mathrm{oMK}} \approx \frac{1}{N} \sum_{i=1}^{N} \varphi\left(Y_v^{\mathrm{OK}} + \sqrt{1 - \left(s_v^{\mathrm{OK}}\right)^2} t_i\right) \tag{28}$$

Another possibility is to use expansions into Hermite polynomials. This approach also enables estimation variance to be expressed (Emery, 2005b, 2006b).

Note that the definition of the ordinary multigaussian kriging estimator makes sense only if the variance of ${Y_v}^{\mathrm{OK}}$ is less than or equal to one. In general, this condition is not severe, but it is not always guaranteed since the kriging weights add to $1/r$; it may be violated if the data are not abundant enough in the kriging neighborhood, for instance if there is only one datum or if the data are highly correlated. In view of these remarks, one must be careful when choosing the kriging neighborhood or when designing the sampling strategy.

Information Effect

The discrete Gaussian model can be extended to the information effect modeling by setting the following hypotheses.

1. The weights $\{\lambda_k, k = 1 \ldots K\}$ in Equation (5) are positive and add to 1. Let $\bar{\mathbf{x}}$ be the random point that coincides with $\mathbf{u}'_k$ with probability λ_k. The random variables $Z_{\bar{\mathbf{x}}}$ and $Y_{\bar{\mathbf{x}}}$ have the same univariate distributions as $Z_{\underline{\mathbf{x}}}$ and $Y_{\underline{\mathbf{x}}}$ respectively and are therefore linked by the same transformation function:

$$Z_{\bar{\mathbf{x}}} = \phi(Y_{\bar{\mathbf{x}}}) \tag{29}$$

2. The block value estimator Z_v^* can be transformed into a Gaussian variable Y_v^* with mean m_v^* and unit variance:

$$Z_v^* = \phi_v^*(Y_v^*) \tag{30}$$

3. The pair $\{Y_{\bar{\mathbf{x}}}, Y_v^*\}$ is bigaussian with correlation coefficient r^*, so that one has:

$$Y_{\bar{\mathbf{x}}} - m = r^*(Y_v^* - m_v^*) + \sqrt{1 - r^{*2}}T \tag{31}$$

   where T is a standard Gaussian variable independent of Y_v^*.
4. The joint distribution of $\{Y_{\underline{\mathbf{x}}}, \underline{\mathbf{x}} \in \mathsf{R}^d\}$, Y_v and Y_v^* is multivariate Gaussian. Henceforth, let r_v be the correlation coefficient of the bigaussian pair $\{Y_v, Y_v^*\}$.

The random variables $Z_{\bar{\mathbf{x}}}$ and Z_v^* are linked by Cartier's relation (Roth and Deraisme, 2001, p. 779). Because of Equations (29) to (31), this implies:

$$\phi_v^*(Y_v^*) = E[\phi(Y_{\bar{\mathbf{x}}})|Y_v^*] = \int \phi(r^*Y_v^* + m - r^*m_v^* + \sqrt{1 - r^{*2}}t)g(t)\,\mathrm{d}t \tag{32}$$

which relates the transformation functions ($\phi_v{}^*$ and ϕ), the correlation coefficient r^* and the means m and $m_v{}^*$. In practice, the latter are supposed to be zero, hence $\phi_v{}^*$ is defined by putting:

$$\phi_v^*(Y_v^*) = \int \phi(r^*Y_v^* + \sqrt{1 - r^{*2}}t)g(t)\,\mathrm{d}t \tag{33}$$

By comparing Equations (32) and (33), it comes:

$$m = r^*m_v^* \tag{34}$$

This relation indicates that, if the mean m of $Y_{\underline{\mathbf{x}}}$ is not exactly zero, then the function $\phi_v{}^*$ defined in Equation (33) transforms $Z_v{}^*$ into a Gaussian variable $Y_v{}^*$ with mean m/r^* instead of 0.

As explained in the introductory section, one wishes to estimate a function of both values Z_v and $Z_v{}^*$, which is also a function of the Gaussian transforms Y_v and $Y_v{}^*$, say $\varphi(Y_v, Y_v{}^*)$. This requires knowing the joint distribution of the pair $\{Y_v, Y_v^*\}$ conditionally to the point-support data. Due to the multivariate Gaussian assumption, this joint distribution is bigaussian:

- the conditional mean is given by the simple kriging estimators Y_v^{SK} and $Y_v^{*\mathrm{SK}}$

- the conditional covariance-variance matrix $\boldsymbol{\Sigma}^{\mathrm{SK}}$ is the covariance-variance matrix of the simple kriging error. After simplification, one finds:

$$\boldsymbol{\Sigma}^{\mathrm{SK}} = \mathbf{R} - \mathbf{S}^{\mathrm{SK}} \tag{35}$$

where $\mathbf{R}$ is the prior covariance-variance matrix of Y_v and Y_v^*; $\mathbf{S}^{\mathrm{SK}}$ is the covariance-variance matrix of Y_v^{SK} and $Y_v^{*\mathrm{SK}}$.

The conditional expectation estimator is therefore:

$$[\varphi(Y_v, Y_v^*)]^{\mathrm{CE}} = \int \varphi\left(\left(Y_v^{\mathrm{SK}}, Y_v^{*\mathrm{SK}}\right) + \sqrt{\mathbf{R} - \mathbf{S}^{\mathrm{SK}}}\,\boldsymbol{t}\right) g(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \tag{36}$$

where $\boldsymbol{t}$ is a standard bigaussian vector with independent components and independent of Y_v^{SK} and $Y_v^{*\mathrm{SK}}$. This estimator is generalized by putting

$$[\varphi(Y_v, Y_v^*)]^{\mathrm{oMK}} = \int \varphi\left(\left(Y_v^{\mathrm{OK}}, Y_v^{*\mathrm{OK}}\right) + \sqrt{\mathbf{R} - \mathbf{S}^{\mathrm{OK}}}\,\boldsymbol{t}\right) g(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \tag{37}$$

which makes sense if $\mathbf{R}$–$\mathbf{S}^{\mathrm{OK}}$ is a positive semi-definite matrix. This estimator is unbiased, irrespective of mean value m of the normal score data. In practice, it can be calculated by drawing many independent standard bigaussian vectors $\{\boldsymbol{t}_1,\ldots \boldsymbol{t}_N\}$ and putting:

$$[\varphi(Y_v, Y_v^*)]^{\mathrm{oMK}} \approx \frac{1}{N}\sum_{i=1}^{N} \varphi\left(\left(Y_v^{\mathrm{OK}}, Y_v^{*\mathrm{OK}}\right) + \sqrt{\mathbf{R} - \mathbf{S}^{\mathrm{OK}}}\,\boldsymbol{t}_i\right) \tag{38}$$

Concerning the derivation of $Y_v^{*\mathrm{OK}}$, note that the kriging weights add to $1/r^*$ so as to ensure unbiasedness, because of relation (34) between the means of Y_v^* and of the normal score data.

Outline of the Method

The following step-wise algorithm is proposed to summarize the discrete Gaussian model approach:

1. Transform the original (declustered) data into standard Gaussian values ($m = 0$); obtain the transformation function ϕ.
2. Model the covariance of the original data on $\{Z_{\mathbf{x}}, \mathbf{x} \in \mathsf{R}^d\}$.

3. Calculate the variance of Z_v and (if information effect is considered) the variance of $Z_v{}^*$ and the covariance between Z_v and $Z_v{}^*$. Determine the correlation coefficients r, r^* and r_v so that the model honors these variances and covariance (Roth and Deraisme, 2001, p. 780; Emery and Soto Torres, 2005, p. 56).
4. Derive the transformation functions ϕ_v and $\phi_v{}^*$ (Eqs. (22) and (33)). In practice, it is convenient to express these functions through expansions into normalized Hermite polynomials (Emery and Soto Torres, 2005, p. 56).
5. Model the covariances of the Gaussian variables $Y_{\underline{\mathbf{x}}}$, Y_v and $Y_v{}^*$. The simple and cross-covariances of $Y_{\underline{\mathbf{x}}}$ and Y_v are determined from the point-support transformation function ϕ, the change-of-support coefficient r and the original data covariance defined in step 2) (Rivoirard, 1994, p. 90). Concerning the cross-covariance between $Y_{\underline{\mathbf{x}}}$ and Y_v^*, a simple approach consists of assuming that it is proportional to the covariance of Y_v (Emery and Soto Torres, 2005, p. 63).
6. Determine the transfer function $\varphi(Y_v,Y_v{}^*)$ to be estimated, e.g. the indirect metal quantity (Eq. (3)).
7. Migrate the point-support data to the centers of the blocks they belong to. From these migrated data and the previously modeled simple and cross-covariances, calculate the ordinary kriging estimators (Y_v^{OK}, $Y_v^{*\mathrm{OK}}$) and the associated covariance-variance matrix $\mathbf{S}^{\mathrm{OK}}$.
8. Simulate a large set of independent Gaussian vectors $\{\boldsymbol{t}_1,\ldots\ \boldsymbol{t}_N\}$, each of them with two uncorrelated components, and estimate $\varphi(Y_v,Y_v{}^*)$ by application of Equation (38).

APPLICATION TO A MINING DATASET

Presentation of the Data

In this last section, the previous methodologies are applied to the local estimation of recoverable reserves in a Chilean porphyry copper deposit. The available data are located in a field of size 400 m × 600 m × 130 m and correspond to a set of 2,376 samples from exploration diamond drill holes composited at 12 m. Figure 1A represents the distribution of the samples intersecting a specific bench of the deposit, together with a grayscale representation of the copper grades. One observes that the highest grades are located in the central part of the bench, which corresponds to tourmaline breccia, whereas the edges of this bench (corresponding to other rock types, mainly granodiorite and other breccias) have smaller grades in general.

Overall, the copper grades have a lognormal-type histogram with an average close to 1.0% Cu (Fig. 1B) and their spatial distribution is anisotropic, with

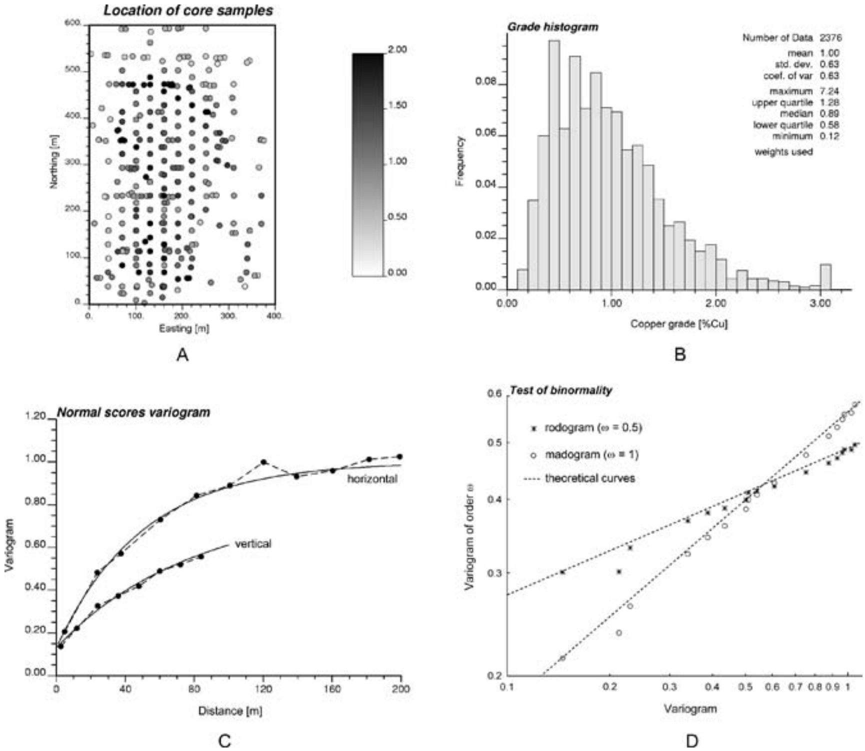

Figure 1. Exploratory and variogram analyses of the copper grade samples: (A) location map of the samples in a specific bench; (B) declustered grade histogram; (C) normal scores variogram along the main anisotropy directions; (D) test of binormality via a comparison of the normal scores variogram with their madogram and rodogram.

a greater continuity along the vertical direction, as shown by a variogram analysis of the normal score transforms (Fig. 1C). The binormality of the two-point distributions is tested by comparing the variogram of the normal score data with their variograms of lower orders (Emery, 2005a): in the present case, the test is satisfactory (Fig. 1D), hence the multigaussian model is deemed suited to the data at hand.

Estimation of the Indirect Metal Quantity

The selective mining units are blocks with size 10 m $\times$ 10 m $\times$ 12 m and the cutoff of interest is equal to 0.5% Cu. At the time of selection, blast holes will be available at a 10 m $\times$ 10 m mesh on each bench of the open pit (which is 12 m

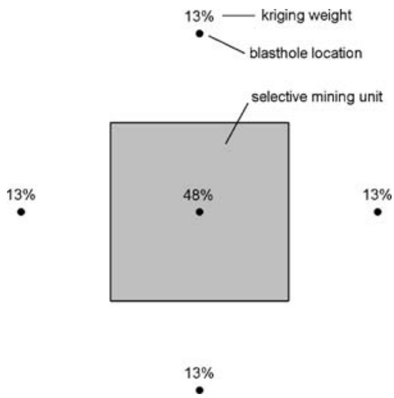

Figure 2. Blast hole configuration showing the locations of the five nearest blast holes in and around the selective mining unit and their ordinary kriging weights. Together with the point-support grade variogram model, these weights allow calculation of the covariance between the true block grade (Z_v) and its estimator ($Z_v{}^*$) (Table 1).

high) and the selective mining unit grades will be estimated by ordinary kriging from the five nearest blast holes (Fig. 2).

One looks for an estimate of the indirect metal quantity (Eq. (3)) that accounts for support and information effects. Six methods are compared:

1. *Conditional expectation* (Fig. 3A) and *ordinary multigaussian kriging* (Fig. 3B) calculated via a Monte Carlo approach (multigaussian model), using the recovery function defined in Equation (8) and the normal score variogram model shown in Fig. 1C. For practical calculations, each selective mining unit is discretized into 16 points ($4 \times 4 \times 1$) and 100,000 random values are used to sample the multivariate normal distribution and perform the numerical integration (Eqs. (12) and (17)).
2. *Conditional expectation* (Fig. 3C) and *ordinary multigaussian kriging* (Fig. 3D) in the framework of the discrete Gaussian model. Following the methodology outlined in the previous section, the original point-support grade covariance and the transformation function ϕ are determined first, then the correlation coefficients r, r^* and r_v are calculated so that the model honors the variance–covariance of Z_v and $Z_v{}^*$ (Table 1). Concerning variogram analysis, the continuous component of the covariance of $\{Y_{\underline{\mathbf{x}}}, \underline{\mathbf{x}} \in \mathsf{R}^d$ and the cross-covariances of the pairs $\{Y_{\underline{\mathbf{x}}}, Y_v\}$ and $\{Y_{\underline{\mathbf{x}}}, Y_v{}^*\}$ are assumed proportional to a single covariance model, which is determined after a transformation of the original point-support grade covariance and is composed of nested cubic structures with greater ranges along the vertical direction.
3. *Ordinary indicator kriging* (Fig. 3E) with an indirect lognormal correction to account for the change of support (Emery, 2004). Nine thresholds are used to derive the conditional distributions, corresponding to the deciles of the global grade distribution. The lower-tail and within-class interpolation are performed thanks to linear models, whereas the upper-tail extrapolation

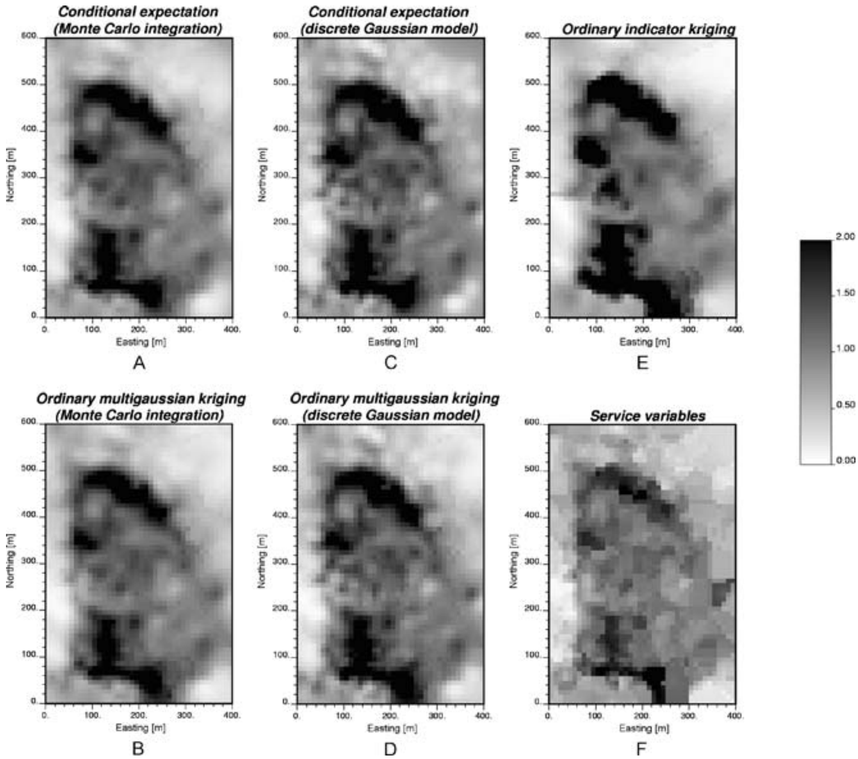

Figure 3. Estimated indirect metal quantities at cutoff 0.5% Cu associated with selective mining units with size 10 m × 10 m × 12 m (representation of a single bench): (A) and (B) conditional expectation and ordinary multigaussian kriging obtained through Monte Carlo integration; (C) and (D) conditional expectation and ordinary multigaussian kriging obtained via the discrete Gaussian model; (E) ordinary indicator kriging; (F) service variables.

Table 1. Parameters for support and information effect modeling

True and estimated grade variances and covariance	$\text{var}(Z_{\underline{\mathbf{x}}})$	0.415
	$\text{var}(Z_v)$	0.298
	$\text{var}(Z_v^*)$	0.300
	$\text{cov}(Z_v, Z_v^*)$	0.295
Correlation coefficients	r	0.874
	r^*	0.849
	r_v	0.957
Variance reduction factor	f	0.723
Slope of the regression of Z_v upon Z_v^*	p	0.983

uses an hyperbolic model with parameter 1.5. To include the information effect, the grade estimator $Z_v{}^*$ is supposed to be conditionally unbiased, so that the problem is to estimate the metal quantity associated with $Z_v{}^*$ instead of Z_v (Eq. (4)). The variance reduction factor is therefore the ratio between the prior variances of $Z_v{}^*$ and $Z_{\mathbf{x}}$, whereas the conditional unbiasedness assumption is corroborated by checking that the slope of the regression of Z_v upon $Z_v{}^*$ is close to 1 (Table 1).

4. *Service variables* (Fig. 3F). This method consists in replacing the point-support data by pseudo data that refer to the quantity to estimate, then an ordinary kriging is applied to these pseudo data (Rivoirard, 1994, p. 100):

$$\forall \alpha \in \{1, \dots n\},\ Z_{\mathbf{x}_\alpha} \rightarrow E\{Z^*_{v_\alpha} I(Z^*_{v_\alpha}; 0.5\%) | Z_{\mathbf{x}_\alpha}\} \tag{39}$$

where v_α stands for the block centered at $\mathbf{x}_\alpha$. Again, conditional unbiasedness for the grade estimator $Z_v{}^*$ is assumed to account for information effect (Eq. (4)).

All the krigings are performed within a moving neighborhood consisting of a maximum of 24 samples. Figure 3 displays the indirect metal quantities at cutoff 0.5% Cu estimated by each method. These estimations do not differ so much from that of the direct metal quantity (not shown here), which indicates that the information effect (effect of ore/waste misclassifications) is quite small, although the grade of each selective mining unit is estimated from five blast hole data only.

In the multigaussian and discrete Gaussian models, ordinary multigaussian kriging provides results that are close to conditional expectation within the sampled area (center of the maps). In contrast, a greater discrepancy is observed in the edges of the maps, in particular when examining the northeast and southeast corners (Fig. 3C and D): the conditional expectation tends to the prior metal quantity (medium gray tone), whereas ordinary multigaussian kriging leads to lower values (light gray tone). The latter is actually more realistic since the edges of the domain do not correspond to ore zones, according to the grades displayed in the sample location map (Fig. 1A) and to the geological knowledge on the deposit (these edges correspond to low-graded granodiorite).

One also notes a difference between the multigaussian and discrete Gaussian models, with a greater smoothing effect for the former (Fig. 3A and C). Such a difference is expected since each approach corresponds to a different set of hypotheses. The discrete Gaussian model loses information about the exact position of the samples within the blocks and is only an approximate model. For instance, if the original data are well described by a multivariate Gaussian distribution, in theory the randomized-location grades cannot follow the same type of distribution. Also, the discrete Gaussian model assumes the permanence of lognormality (Rivoirard, 1994, p. 82), which is not true in rigorous sense. This situation is the price to be paid for looking for a simple analytical solution to the change-of-support problem. In contrast, the multigaussian approach with Monte Carlo integration is theoretically sound and needs less assumptions, but CPU requirements are significantly high. Both models are worthy options for evaluating recoverable reserves in ore deposits: the differences between the multigaussian and discrete Gaussian models are not so important when compared to the other two methods (indicator kriging and service variables), which clearly lack precision with respect to conditional expectation. In the case of service variables, this may be explained by the fact that the method is based on indirect and somehow incomplete information (Eq. (39)). As far as indicator kriging is concerned, the reasons for loss of precision may be the loss of information due to the indicator coding of the original data and to the individual estimation of each indicator, the sensitiveness to the post-processing of the point-support local distributions (especially in what refers to tail extrapolation) and the difficulties associated with change-of-support correction (Emery and Ortiz, 2004).

CONCLUSIONS

Two approaches have been presented to account for support and information effects when estimating transfer or recovery functions: (a) Monte Carlo integration using the point-support multigaussian model, and (b) the discrete Gaussian model, which is simpler and faster to use but requires further hypotheses. In each approach, ordinary multigaussian kriging provides an unbiased estimate of recovery functions that does not use the mean value of the normal score data. Furthermore, the estimators are close to the optimal conditional expectation when the data are abundant and are not dominated by the normal score global mean when the data are scarce. Thereby, they appear as promising alternatives for estimating recoverable reserves in ore deposits, especially if this mean may have been misspecified or is deemed unsuitable at local scale.

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