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# Anisotropic viscoelastic models with singular memory

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**Abstract**

The physical background of singular memory models and in particular the Cole–Cole model is discussed. Three models of anisotropic linear viscoelasticity with frequency-dependent stiffness coefficients are considered. The models are constructed in such a way that anisotropic properties are separated from anelastic effects. Two of these models represent finite-speed wave propagation with singularities at the wavefronts (the exponential relaxation model) and without singularities at the wavefronts (the Cole–Cole model), while a third model called the fractional model is related to the constant $Q$ with unbounded propagation speed. The Cole–Cole and fractional models belong to the class of singular memory models studied earlier because of their applications in polymer rheology, poroelasticity, poroacoustics, seismic wave propagation and other applications. Well-posedness of initial boundary value problems with mixed Dirichlet–Neumann boundary conditions is established for the three models. Regularity properties of the three models are examined.


*Keywords:* Viscoelasticity; Anisotropy; Attenuation

## 1. Introduction

The Cole–Cole hereditary law has been very successful in modeling dispersive properties of dielectrics (Cole and Cole, 1941) and anelastic mechanical response in polymers, where it is known by the name of Bagley–Torvik (Bagley and Torvik, 1986). It allows a very accurate modeling of the frequency dependence of the dielectric constant or the anelastic moduli over a very large range of frequencies provided the material exhibits only a single transition region between the low-frequency and the high-frequency regime (and a single minimum of the $Q$ parameter) (Soula and Chevalier, 1998). It is known from alternative models of the same response function that a comparable result based on rational functions requires polynomials of fifth degree (Soula et al., 1997). It therefore comes as no surprise that the Cole–Cole model has recently been found to allow a very good matching for the observed dispersive properties of elastic moduli in rocks (Batzle et al., 2001). In a related context of dielectric properties of porous media it appears in Ruffet et al. (1991) and Taherian et al. (1990). The Cole–Cole model has a potential for adding new features, such as additional transition zones in the spectral response function, by including additional fractional powers of frequency in the numerator and denominator (Torvik and Bagley, 1983; Fenander, 1997). Dispersion in rocks, mostly due to the fluid saturation, generally exhibits a single transition zone and a single peak of the inverse quality factor (e.g., Tutuncu et al., 1998). The Cole–Cole model in its simplest form is adequate for this case.

The Cole–Cole model is a member of a class of models which are defined in terms of “springpots” (Koeller, 1984), which constitute an intermediate ob-

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ject between a spring and a dashpot, in the way of Maxwell–Voigt–Zener models. The springpots are parameterized by their order, contained between 0 and 1, with 0 corresponding to the spring and 1 corresponding to the dashpot. A Kelvin–Voigt element with a springpot replacing the dashpot is often used as a model of a viscous oscillation damper (for an extensive bibliography on this subject and other engineering applications, see Agrawal (1999); Rossikhin and Shitikova, 1997). Since springpots can have different orders, this class of models is much richer than the Maxwell–Voigt class. For example replacing a single dashpot by two springpots of different order connected in parallel (Rossikhin and Shitikova, 2001) yields a new model of viscous response. In this context the Cole–Cole relaxation is a generalized Standard Linear Solid with the dashpots replaced by springpots of the same order.

The Cole–Cole model belongs to the category of singular memory models, i.e., hereditary models with the constitutive relations represented by weakly singular convolution kernels. The first singular memory model of viscous behavior was due to Boltzmann (Boltzmann, 1876). It involved the Hilbert kernel $(t-\tau)^{-1}$. It is nowadays known that such a strong (nonintegrable) singularity results in an infinite speed of propagation (Renardy, 1982).

In hereditary viscoelasticity the stress is given by the constitutive law $\sigma(t) = G(0)\, e(t) + G'(t) * e(t)$. In general, $G(0) > 0$ ensures a finite propagation speed. Singular memory is defined by the asymptotic property $G'(t) \sim \text{const} \times t^{-\gamma} + \ldots$ for $t \to 0$, $0 < \gamma < 1$, of the kernel. Note that $G(t)$ is the stress relaxation function (stress response to a jump of strain from zero to a constant level at one). The singularity of the delayed response kernel $G'(t)$ gives much more weight to the very recent past of the strain history than to the rest of it.

The Cole–Cole model can also be represented in terms of a relaxation equation for stress and strain involving fractional derivatives (Podlubny, 1998; Miller and Ross, 1993). More general relaxation laws involving fractional derivatives have been discussed by Bagley and Torvik (1983c), Torvik and Bagley (1984) and Fenander (1997). Relaxation models based on fractional derivatives originated in the theory developed by Gemant (1936, 1938) and Blair (1944). The theory of Gemant and Scott Blair was in turn stimulated by the papers of Nutting on generic creep laws (Nutting, 1921, 1943, 1946), who pointed out that the experimental creep laws of that time could be expressed by a simple relation between the strain, stress and time $e \sim \sigma^{\alpha} t^{\beta}$, with $\alpha > 0$ accounting for nonlinearity and $0 < \beta \leq 1$ indicating that a fractional derivative of order $< 1$ may be involved. In Russian literature two classes of weakly singular kernels related to the Cole–Cole law, the Rabotnov kernel (Rabotnov, 1969, 1980) and the Rzhanitsyn kernel (Rzhanitsyn, 1949, 1968; Rossikhin and Shitikova, 1997), are widely used. Applications of fractional derivatives in phenomenological polymer rheology and in disordered systems have a long and rich history (Bronskii, 1941; Slonimsky, 1939, 1961, 1967; Havriliak and Negami, 1967; Nigmatullin, 1984, 1986; Friedrich, 1991; Friedrich and Braun, 1992; Bagley and Torvik, 1983a, 1986; Torvik and Bagley, 1983; Glöckle and Nonnenmacher, 1991, 1994; Schiessel and Blumen, 1993; Heymans and Bauwens, 1994; Rogers, 1983; Eldred et al., 1995; Palade et al., 1996), to mention but a few representative papers. Another material often reported in this connection was rubber, for which a simple viscous memory kernel $(t-\tau)^{-\gamma}$, $0 < \gamma < 1$, was suggested by Duffing (1931). De Andrade's law for creep in metals also falls in this category (Andrade, 1910, 1912).

Power laws of attenuation in acoustics (Szabo, 1994, 1995; Szabo and Wu, 2000) and in seismology (Strick, 1984) can be expressed in terms of fractional derivatives The corresponding theoretical models are discussed in Hanyga and Seredyńska (1999a, 2002), Hanyga and Rok (2000) and in some earlier papers referenced there. More recently, evidence for the validity of power law has come from inversion of laboratory data and marine seismic data (Ribodetti and Hanyga, 2004). Although power laws can be considered as a high-frequency approximation of the Cole–Cole model, they do not satisfy the requirements of the theory developed in this paper in the anisotropic case.

In engineering, fractional models of viscoelastic relaxation were applied among others by Enelund et al. (1997), Enelund and Josefson (1997), Enelund and Lesieutre (1999), Enelund and Olsson (1998), Bagley (1990), Bagley and Torvik (1983b), Torvik and Bagley (1985) Bagley and Calico (1991) and Schmidt et al. (2000). In the general context of viscous mechanical damping of structural elements in particular the references Suarez and Shokooh (1995, 1997), Beyer and Kempfle (1995), Gaul et al. (1989, 1991), Mbodje et al. (1994) and Seredyńska and Hanyga (2000) make use of fractional derivatives. In rock mechanics the Cole–Cole elastic modulus was discussed by Jones (1986). Additional references can be found in Podlubny (1998) and Rossikhin and Shitikova (1997).

One of the first applications of fractional calculus in viscoelasticity was, however, seismic attenuation (Caputo and Mainardi, 1971, 1976; Caputo, 1969), in particular the Caputo–Kjartansson constant $Q$ model (Caputo, 1967a,b; Mainardi and Tomirotti, 1997; Hanyga, 2002b; Carcione et al., 2002; Kjartansson, 1979). In the same context the Cole–Cole model was discussed and compared with the Caputo–Kjartansson model by Jones (1986). In a scalar model of wave propagation, the constant $Q$ can be modeled by a Cole–Cole law with a low value of the exponent (Hanyga, 2003a). Unlike the Caputo–Kjartansson model, the Cole–Cole model is compatible with the requirement of finite propagation speed. In rock mechanics dispersion and attenuation is essentially due to pore fluid flow (Tutuncu et al., 1998) and can thus be accounted for by poroelastic models. Fast-to-slow wave conversion (Gurevich and Lopatnikov, 1997; Hanyga and Rok, 2000), squirt flow (Dvorkin and Nur, 1993) and patchy fluid saturation (Johnson, 2001) can be expected to generate singular memory effects in the seismic frequency range. Biot's theory also involves a weakly singular kernel associated with the dynamic permeability (Hanyga, 1999a, 2001c). In Biot's poroelasticity, singular memory effects are expected be observable above the Biot frequency, i.e., in the acoustic log and ultrasound frequency ranges. For rigid air-saturated materials, both the permeability and bulk modulus are represented by singular memory kernels (an extensive bibliography and mathematical analysis can be found in Hanyga (2001b)).

Poroelasticity (e.g., Norris, 1986; Berryman et al., 1988) and poroacoustics (e.g., Wilson, 1992; Allard and Champoux, 1992; Stinson and Champoux, 1992) involve singular memory kernels of a more complicated type. Singular memory models of anelastic mechanical response constitute a much larger class than fractional derivative models. In wave propagation, modeling the former can often be replaced by the latter. Asymptotic analysis of wave propagation shows that the singular part of the memory kernel plays a predominant role in shaping the signal of the wavefront region, while the regular part of the memory merely accounts for an additional amplitude decay and can often be replaced by a constant (Hanyga and Seredyńska, 1999a, 2002). Consequently, relaxation models involving singular kernels can be simplified by introducing simple fractional relaxation laws with the regular part of the memory kernel replaced by a constant.

In scalar hyperbolic problems the memory effects are represented by invertible convolution operators. Assuming a homogeneous medium memory, operators can be flipped from the spatial derivative operator to the inertial terms (time derivatives) and vice versa. Applying the simplification described in the previous paragraph to an equation in which the memory appears in the inertial term, one obtains a solution which represents a generalized power law of attenuation (possibly with several power terms involving different exponents; Hanyga and Seredyńska, 2002). An equivalent equation with the memory operator acting on the space derivatives is more complicated and involves generalized Mittag–Leffler functions (Engler, 1997). Such equations have some explicit solutions Hanyga and Seredyńska (1999a, 2002) that can be studied in much detail. Unfortunately, these special models have some undesirable features, e.g., an unbounded creep function (Engler, 1997). It can, however, be shown that the power law behavior is a universal feature of asymptotic solutions for singular memory models, i.e., it is a general feature of the spectrum of the signal following a wavefront (Hanyga and Seredyńska, 1999a, 2002).

The Cole–Cole relaxation model has many advantages over the other models. It is compatible with a finite speed of propagation. It essentially involves only the singular response and therefore can be represented by fractional derivatives. The Cole–Cole modulus has the same form whether applied to the inertial term or to the spatial term in the equation. The Cole–Cole creep function is bounded and tends to the inverse relaxed modulus for $t \rightarrow \infty$. Constant $Q$ can be simulated by a Cole–Cole model with a low value of the exponent (Hanyga, 2003a).

In a hyperbolic model the speed of propagation is bounded by its infinite frequency limit. The solution vanishes outside a region bonded by a wavefront. Singular memory entails smoothing at the wavefronts, as shown in various contexts in (Narain and Joseph, 1982; Renardy, 1982; Hrusa and Renardy, 1988; Desch and Grimmer, 1986, 1989a,b; Prüss, 1993; Hanyga, 2001b). Combination of finite wave propagation speed with smoothing results in a signal delay with respect to the wavefront (Hanyga and Seredyńska, 1999b; Hanyga and Rok, 2000). These effects were studied in much detail in the case of scalar hyperbolic equations with singular memory (Lokshin and Rok, 1978a,b; Hanyga and Rok, 2000; Hanyga, 2001b; Hanyga and Seredyńska, 2002), where explicit solutions can be often constructed. The solutions of such scalar hyperbolic equations largely overlap with phenomenological power-frequency models of acoustic attenuation (Szabo, 1994, 1995; Szabo and Wu, 2000; Strick, 1984). Simple attenuation models such as power laws or constant $Q$ can only be obtained in the context of scalar equations. Since simplicity is lost in anisotropic models, flexibility of the Cole–Cole and related models becomes a more important asset.

For nonlinear viscoelastic problems with $G(0)>0$ (i.e., with a bounded propagation speed) and with singular memory, smoothing does not extend to shock waves, which propagate as discontinuities (Gripenberg, 2001).

Smoothing in a hyperbolic model entails an additional effect: signal delay with respect to the wavefront. The wavefront is the boundary between the undisturbed medium and the wavefield. Its speed of propagation is equal to the infinite frequency limit of the frequency-dependent propagation speed. A smooth peaked signal must lag behind the wavefront because the only peaked signals that can propagate with the wavefront are the Dirac delta and its derivatives. The resulting signal delay for a source delta-spiked signal can be calculated (Hanyga and Seredyńska, 1999b, 2002).

In this paper the Cole–Cole stress relaxation law will be incorporated in the constitutive relations of anisotropic elasticity following a method of Carcione and Cavallini (1994) and Carcione et al. (1996). The method of Carcione and Cavallini (1994) and Carcione et al. (1996) is based on an a priori constraint on the constitutive equations which restricts the frequency dependence to the eigenvalues of the stiffness tensor. Directional dependence of anelastic medium parameters is represented by frequency-independent eigenstrain tensors. The material symmetry of the medium and the directional dependence of anelastic moduli are thus decoupled from the frequency-dependent scalar parameters. Such an assumption may be inconsistent with physical models derived by averaging, homogenization or physical models (Gurevich, priv. comm). In the absence of a physical model it is, however, a convenient tool for restricting the number of model parameters and ensuring causality. In seismology, information about the anisotropic features of the medium is obtained on the assumption that the medium is elastic. It is therefore convenient to preserve this information while introducing viscous effects. The decomposition of stiffness matrix in terms of eigenstrains and scalar stress relaxation kernels is, however, not necessary for the well-posedness and regularity, as shown in Hanyga (2003c).

Regularity and well-posedness results obtained so far apply to some special classes of equations (one-dimensional viscoelasticity and poroacoustics (Hanyga, 2001b), or anelastic models with a single relaxation mechanism (Prüss, 1993; Desch and Grimmer, 1986)). In particular, existence, uniqueness and regularity results for three-dimensional anelastic models with several different relaxation functions controlling the stress response have not been obtained previously. We present here the uniqueness, existence and regularity results obtained recently in Hanyga (2003c), skipping mathematical details and focusing on the Cole–Cole relaxation model and on two qualitatively different relaxation models, viz., the exponential relaxation and the fractional relaxation.

It should be emphasised that in hereditary viscoelasticity the issue of existence and uniqueness is far from trivial and depends on such properties of the memory kernels as positivity, monotonicity and convexity (Hrusa and Renardy, 1988; Desch and Grimmer, 1989a,b; Prüss, 1993). For the kernels considered in this paper a stronger property (complete monotonicity) was more convenient in proofs of existence and uniqueness. Regularity of solutions and finite propagation speed depend on additional properties of these kernels (Section 5). As mentioned above, finite propagation speed in combination with smoothness implies signal delay. In the context of the relaxation models discussed in this paper only the Cole–Cole relaxation has both features. Relevance of signal delay for inversion is demonstrated in Hanyga and Seredyńska (1999b).

Numerical methods for solving basic initial-value problems for the viscoelastic media with the Cole–Cole relaxation kernel and other singular memory kernels are discussed in Rossikhin and Shitikova (2000) and Hanyga (2002a). They are based on a reformulation of the problem in terms of fractional derivatives and an adequate discretization. Time-stepping schemes for fractional time derivatives have been developed (Hanyga, 1999b,c) and implemented (Carcione et al., 2002) in a scalar model of constant $Q$ due to Caputo and Mainardi (Caputo, 1967a; Mainardi and Tomirotti, 1997). The earliest algorithms for fractional time-stepping in structural mechanics are due to Padovan (1987) (with FEM), Koh and Kelly (1990) and Makris et al. (1993) (with BEM). Alternatively, it is possible to express the problem in terms of partial differential equations with additional variables satisfying fractional relaxation equations (Hanyga, 2003b). Important applications in polymer rheology have led to a recent spate of numerical schemes for ordinary differential equations of fractional order designed for applications in viscoelastic and viscoplastic models (the references Diethelm et al. (2002) and Diethelm and Freed (1999) are most relevant for the equations considered in Hanyga (2003b)). It is also possible to eliminate fractional derivatives by introducing an auxiliary variable satisfying an integrodifferential equation (Mbodje et al., 1994; Yuan and Agrawal, 1998). General multistep schemes of an arbitrary order of accuracy can be derived from Lubich's quadratures and numerical schemes for Volterra equations (Lubich, 1986, 1988a,b; Brunner and van der Houwen, 1986).

## 2. The Cole–Cole response

The Cole–Cole transfer function can be expressed in the following form

$$\hat{M}(\omega) = M_\infty \frac{1 + a(-i\omega\tau)^{-\alpha}}{1 + (-i\omega\tau)^{-\alpha}} \tag{1}$$

where $\hat{M}(\omega)$ stands for some frequency-dependent modulus (e.g., Young's modulus in Bagley and Torvik's polymer model), $M_\infty$ denotes the high-frequency limit of $\hat{M}(\omega)$, $\tau$ represents a characteristic relaxation time and $0 < \alpha < 1$. Thermodynamic arguments (Bagley and Torvik, 1986) imply that the exponent $\alpha$ in the numerator and in the denominator are equal. Furthermore, the relaxed modulus $M_0 = aM_\infty$ cannot exceed the instantaneous modulus $M_\infty$, hence $a < 1$. This inequality follows from thermodynamics (Fabrizio and Morro, 1992; Bagley and Torvik, 1986) and is known from creep tests. The exponent $\alpha$ controls the width of the transition zone between $M_0$ and $M_\infty$.

The asymptotic behavior of the Cole–Cole transfer function for $\omega \to \infty$

$$\hat{M}(\omega) \sim M_\infty [1 + (a-1)(-i\omega\tau)^{-\alpha}] \tag{2}$$

The first term corresponds to an immediate elastic response. The delayed response is dominated by a power law, which is a well-known universal law for relaxations, including mechanical relaxation (Jonscher, 1996). Universality of this feature has prompted many scientists to develop a statistical model of relaxation based on direct statistical arguments (Weron and Kotulski, 1996) or on anomalous diffusion, more specifically, on the continuous time random walk model of anomalous diffusion (Gomi and Yonezawa, 1995). The relaxation of the modes of the associated fractional Fokker–Planck equation turns out to obey the Cole–Cole law (sometimes referred to as the Mittag–Leffler law) (Metzler et al., 1999). An important feature of these theoretical models is universality due to a very fundamental and generic physical micromodel.

In time domain the Cole–Cole response is expressed in terms of a time convolution. Let $E_{\gamma,\beta}(z)$ denote the generalized Mittag–Leffler function (Podlubny, 1998)

$$E_{\gamma,\beta(z)} := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n + \beta)} \tag{3}$$

for arbitrary complex $z$. The Mittag–Leffler function proper $E_\gamma(z) := E_{\gamma,1}(z)$ is a special case of the generalized Mittag–Leffler function.

The stress relaxation function of the Cole–Cole model is

$$R(t) = M_0 + (M_\infty - M_0)E_\alpha(-(t/\tau)^\alpha) \tag{4}$$

(Fig. 1) (Hanyga, 2002a).

For a strain $e(t)$ such that $t^\alpha e(t) \to 0$ for $t \to 1$ and $e$ is right-continuous at 0, the stress is given by inverse Fourier transform of $\hat{M}(\omega)\ \hat{f}(\omega)$

$$\sigma(t) = M_\infty e(t) + (1-a)M_\infty \times \int_0^\infty E_\alpha(-(\theta/\tau)^\alpha) e'(t-\theta)\mathrm{d}\theta \tag{5}$$

in the time domain.

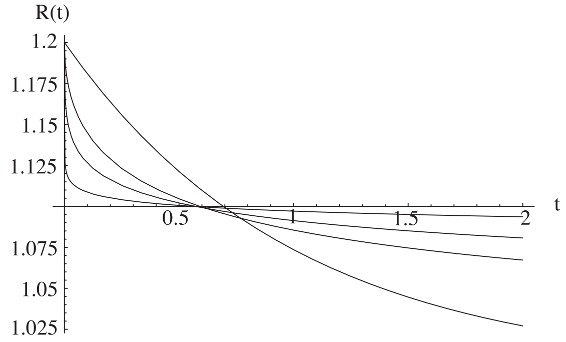


Fig. 1. The Cole–Cole relaxation function $R(t)$ for $\alpha = 0.5$, 0.3, 0.1 compared with exponential relaxation $R_{\exp}(t) = M_\infty + (M_0 - M_\infty)\exp(-t/\tau)]$. The Cole–Cole relaxation function has an infinite slope at $t = 0$. Smaller values of $\alpha$ correspond to higher initial slope and longer tails. $M_0 = 1$, $M_\infty = 0.8\, M_0$.

The Laplace transform $\tilde{K}$ of the corresponding compliance $K(t)$ is given by the equation $\tilde{R}(s)\tilde{K}(s) = 1/s$, hence

$$\tilde{K}(s) = M_\infty^{-1} \frac{1 + a^{-1}(s\tau')^{-\alpha}}{1 + (s\tau')^{-\alpha}}$$

where $\tau' := a^{-1/\alpha}\tau$. The creep function $J(t)$, i.e., the strain response to a unit step stress jump, is the inverse Laplace transform of $s^{-1}\tilde{K}(s)$, hence $J(t) = M_\infty^{-1} + (M_0^{-1} - M_\infty^{-1})E_\alpha(-(t/\tau')^\alpha)$ (Fig. 2).

The corresponding strain rate has the singularity $M_\infty^{-1}(a^{-1} - 1)\tau'^{-1}(t/\tau')^{\alpha-1}$, which implies that the initial creep rate is infinite, a well-known fact in experimental mechanics (cf. e.g., Kohlrausch, 1847, 1863; Nutting, 1946; Rabotnov, 1969; Rossikhin and Shitikova, 1997).[1] Attempts to incorporate this feature in the stress relaxation function (or its counterpart the dielectric decay function) has led to the stretched exponential a.k.a. Kohlrausch–Watts–Williams (KWW) law $G(t) = M\exp(-(t/\tau)^\alpha)$, $0 < \alpha < 1$ (see Williams and Watts, 1970). The stretched exponential is an empirical law with a rather complicated frequency domain counterpart (Williams and Watts, 1970). The Cole–Cole response has the same short-time asymptotics as the (KWW) law and a simple algebraic frequency-domain form, convenient for parameter estimation. The two differ in the long time range: in contrast to the KWW law, the Cole–Cole response has a fat tail (an algebraic decay rate at infinity). The last feature has little relevance for wave propagation but there is an experimental evidence for it in polymer rheology (Klafter et al., 1986).

Unlike some other fractional derivative models, the Cole–Cole relaxation model is characterized by a bounded creep function $J(t)$. In fact

$$\tilde{J}(s) = \frac{1}{M_0 s}\frac{1 + (\tau s)^\alpha}{a + (\tau s)^\alpha} \sim \frac{1}{asM_0}[1 - (a^{-1} - 1)(\tau s)^\alpha]$$

for small $s$. Hence, taking into account the nonnegativity and monotonicity of the creep function, we have by a Tauberian theorem

$$J(t) \sim \frac{1}{M_\infty}[1 - (a^{-1} - 1)(t/\tau)^{-\alpha}]$$

tends to $1/M_\infty$ for $t \to \infty$.

The Cole–Cole response has some statistical justification (Gomi and Yonezawa, 1995; Weron and Kotulski, 1996) of a rather universal nature.

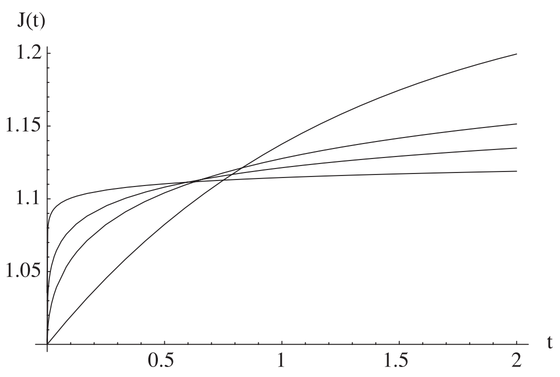


Fig. 2. The corresponding creep functions $J(t)$. Lower values of $\alpha$ correspond to higher initial train rate and slower final strain rate.

[1] In a hyperbolic model of viscoelasticity the creep function $J(\tau)$ and the stress relaxation function $G(\tau)$ have identical singularities. Indeed, the relations $\sigma = G * e'$ and $e = J * \sigma'$ imply that their Laplace transforms satisfy the relation $\tilde{J}(s)\,\tilde{G}(s) = s^{-2}$. For a singular memory hyperbolic model with $G(t) = G(0)H(t) - at^\alpha/\Gamma(\alpha+1) + \ldots$, $0 < \alpha < 1$, where $H(t)$ denotes the unit step function, an Abelian theorem implies that $\tilde{G}(s) \sim G(0)\,s^{-1} - a\,s^{-\alpha-1} + \ldots$ for $s \to \infty$, whence $\tilde{J}(s) \sim G(0)^{-1}\,s^{-1} + [a/G(0)^2]\,s^{-\alpha-1}$ in the same limit. In view of the positivity and monotonicity property a Karamata Tauberian theorem (Widder, 1946) implies that $J(t) = G(0)^{-1}\,H(t) + [a/G(0)^2]\,t^\alpha/\Gamma(\alpha+1) + \ldots$ for small $t$.

## 3. The anisotropic Cole–Cole model

The elastic stiffness tensor $c_{klmn}$ can be considered as a $6\times 6$ positive semidefinite symmetric matrix $C_{rs}$. The matrix $C_{rs}$ can be decomposed in terms of dyadics $v_r^{(p)} v_s^{(p)}$, where $v^{(p)}$ denote the unit eigenvectors of $C_{rs}$. Reverting to tensor notation for the three-dimensional space

$$c_{klmn} = \sum_{p=1}^{6} \Lambda_p^0 e_{kl}^{(p)} e_{mn}^{(p)} \tag{6}$$

where $\Lambda_p^0$, $p=1,\ldots 6$, are the eigenvalues of $C_{rs}$ and the eigenstrains $e_{kl}^{(p)}$ are the eigenvectors $v_r^{(p)}$ expressed in $3\times 3$ tensor notation. They are assumed orthonormal in the sense of

$$e_{kl}^{(p)} e_{kl}^{(q)} = \delta_{pq} \tag{7}$$

(Helbig, 1994). An eigenstrain is defined by the property that the corresponding stress differs from it by a scalar factor $\Lambda$. We shall use the shorthand notation $E=\{e_{kl}\}$, $E^{(p)}=\{e_{kl}^{(p)}\}$, $\langle E,E'\rangle = e_{kl}\, e'_{kl}$, $C[E]=\{g_{klmn}\, e_{mn}\}$ and $C=\sum_p \Lambda_p^0 E^{(p)}\otimes E^{(p)}$ and fixed, frequency-independent material symmetry properties.

We shall now construct a viscoelastic memory kernel $G(t)$ with the elastic part $G(0)=C$. Let $\mathcal{G}$ denote the material symmetry group of the elastic medium defined by the stiffness tensor $C$. For a solid in a nondeformed configuration (Truesdell, 1972) $\mathcal{G}\subset SO(3)$ and the property $Q\in\mathcal{G}$ is equivalent to $C[\hat{Q}[E]]=\hat{Q}[C[E]]$ for every symmetric tensor $E$, where $\hat{Q}[E]:=QEQ^{T}$ is a rotation in the six-dimensional space of symmetric tensors $E$, and

$$C[E]:=\sum_{p=1}^{6} \Lambda_p^0 \langle E, E^{(p)}\rangle E^{(p)} \tag{8}$$

The eigenvalues $\Lambda_p^0$ can be divided into groups in such a way that $\Lambda_p^0$ and $\Lambda_r^0$ are equal if and only if they belong to the same group. We now associate the parameters $a_p$, $\alpha_p$, $\tau_p$, $0<\alpha_p<1$, $0\le \alpha_p \le 1$, $\tau_p>0$, with each group of eigenvalues and define

$$\Lambda_p(-i\omega):=\Lambda_p^0 \frac{1+a_p(-i\omega\tau_p)^{-\alpha_p}}{1+(-i\omega\tau_p)^{-\alpha_p}} \tag{9}$$

The tensor-valued anelastic stress relaxation function is now given by the formula

$$\hat{G}(\omega) = \sum_p (-i\omega)^{-1}\Lambda_p(-i\omega)E^{(p)}\otimes E^{(p)} \tag{10}$$

or, in the time domain,

$$G(t) = \sum_p r_p(t) E^{(p)}\otimes E^{(p)} \tag{11}$$

where $r_p(t)$ denotes the inverse Laplace transform of $\Lambda_p(s)/s$. The corresponding stress–strain response function $\lambda_p(t)=r_p'(t)$ is given by Eq. (5). According to Theorem 2 in Appendix 2 of Hanyga (2002a), every $Q\in\mathcal{G}$ is a material symmetry of $\hat{G}(\omega)$ and vice versa, i.e., $\hat{G}(\omega)$ and $C$ have the same material symmetry for every $\omega$. Thermodynamic restrictions on $G$ (Fabrizio and Morro, 1992) are satisfied in view of the inequality $a_p\le 1$ in combination with the major symmetry of $G_{klmn}(t)$.

For comparison we shall also discuss the exponential relaxation and the fractional relaxation model:

$$\Lambda_p(s) = a_p + \sum_{n=1}^{N} b_p^{(n)}/(s+1/\tau_p^{(n)});$$

$$\lambda_p(t) = a_p\delta(t) + \sum_{n=1}^{N} b_p^{(n)} e^{-t/\tau_p^{(n)}} \tag{12}$$

$$\Lambda_p(s) = a_p s^{\alpha_p};\quad \lambda_p(t) = a_p t^{-\alpha_p-1}/\Gamma(-\alpha_p) \tag{13}$$

with $a_p>b_p\ge 0$ in the first case and $a_p>0$ in the second case. The last inequality in combination with the major symmetry of the tensor-valued function $G$ ensures that the thermodynamic restrictions are satisfied (Fabrizio and Morro, 1992).

## 4. Energy dissipation

In the absence of thermal effects the condition of nonnegative dissipation usually takes the form

$$\psi' \le \langle \sigma, e'\rangle$$

where $\psi$ is some sort of energy. Unfortunately, the energy is not uniquely defined in a hereditary medium and, consequently, it is convenient to consider energy dissipation on a cycle (or on a quasi-cycle, since cycles consistent with the equations of motion may not exist in a sufficient number):

$$\oint \langle \sigma(t), e'(t) \rangle \mathrm{d}t \equiv \int_0^T \langle \sigma(t), e'(t) \rangle \mathrm{d}t \geq 0 \tag{14}$$

for periodic $e(t)$ with a period $T$ (in this case $\sigma(t)$ is clearly periodic too). Inequality (14) entails the inequalities $G_\infty := \lim_{t \to \infty} G(t) \geq 0$ and Graffi's inequality

$$\int_0^\infty G(t)\sin(\omega t)\mathrm{d}t < 0 \qquad \text{for } \omega > 0$$

As shown in Fabrizio and Morro (1992), this in turn implies that

$$\int_{-\infty}^t \langle \sigma(\tau), e'(\tau) \rangle \mathrm{d}\tau \geq 0 \tag{15}$$

for arbitrary square integrable $e(t)$ having square integrable derivatives.

Obviously, Eq. (15) entails Eq. (14) as a particular case. This provides us with a criterion of thermodynamic dissipativity.

We shall now show that Eq. (15) is satisfied in virtue of our assumptions in the previous section.

Let $e$ be a constant symmetric matrix. Since

$$\langle e, G(t)e \rangle = \sum_p r_p(t) \langle e, E^{(p)} \rangle^2$$

the derivatives of $G(t)$ satisfy the inequalities

$$(-1)^n \langle e, G^{(n)}(t)e \rangle \geq 0 \qquad n = 0, 1, 2, \ldots \tag{16}$$

for $t>0$. In the terminology of Gripenberg et al. (1990), $G(t)$ is a completely monotonic matrix function on $t>0$ and, consequently, it is infinitely differentiable for $t>0$. In particular, Eq. (16) restricted to $n=0,1,2$ implies that $G(t)$ is a matrix function of positive type (Gripenberg et al., 1990; Yosida, 1974), whence

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \langle \varepsilon(t), G(t-\tau)\varepsilon(\tau) \rangle \mathrm{d}t\mathrm{d}\tau \geq 0 \tag{17}$$

for arbitrary square integrable functions $\varepsilon(t)$ with values in the space of symmetric matrices. Taking in particular

$$\varepsilon(t) = \begin{cases} e'(t), & t < T \\ 0 & t > T \end{cases}$$

for an arbitrary $T>0$, we have, in view of causality,

$$\int_{-\infty}^T \mathrm{d}t \int_{-\infty}^t \langle e'(t), G(t-\tau)e'(\tau) \rangle \mathrm{d}\tau \equiv \int_{-\infty}^T \langle \sigma(t), e'(t) \rangle \mathrm{d}t \geq 0$$

hence, Eq. (15) is satisfied on account of $(-1)^n r_p^{(n)}(t) \geq 0$ for $t>0$ and $n=0,1,2$.

This explains the role of monotonicity and convexity of memory functions in hereditary viscoelasticity.

## 5. Well-posedness

In view of their complicated form integrodifferential equations require a verification of the well-posedness of the associated initial-boundary value problems. The existence and uniqueness of solutions depend on the monotonicity and convexity of the memory kernel.

The initial-boundary value problem for the equations of motion will be formulated along the lines of Desch and Grimmer (1989a). Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$-smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $n_k$ denote the unit exterior normal on $\Gamma_2$. We shall consider the boundary-value problem on $\mathbb{R} \times \Omega$

$$\rho \dot{v}_k = \sigma_{kl,l} \qquad \text{for } k = 1, 2, 3 \tag{18}$$

$$\sigma_{kl} = g_{klmn} * v_{m,n} \tag{19}$$

$$v_k(t, \boldsymbol{x}) = 0, \;\; k = 1, 2, 3 \quad \text{on } \Gamma_1$$

$$\sigma_{kl}(t, \boldsymbol{x}) n_l = 0, \;\; k = 1, 2, 3 \quad \text{on } \Gamma_2 \tag{20}$$

with initial data derived from an initial history $v_k^{(0)}(t, \boldsymbol{x})$, $k=1,2,3$, $t \leq 0$, of the velocity field.

For the proof of well-posedness the key property of the matrix-valued memory kernel $G$ is complete monotonicity, defined here by the equation

$$G(t)=\int_0^{\infty} e^{-t\zeta}\mathrm{d}F(\zeta), \qquad t>0 \tag{21}$$

where $F=\{f_{klmn}\}_{k,l,m,n=1,2,3}$ is a tensor-valued function on $\overline{\mathbb{R}_+}$ with the symmetries of $g_{klmn}$

$$F(\zeta)=\sum_{p=1}^{P}\mu_p(\zeta)G^{(p)} \tag{22}$$

$\mu_p$, $p=1,\ldots,P$, are nondecreasing functions with bounded variation, the integral is interpreted as a Laplace–Stieltjes integral and the tensors $G^{(p)}$ are positive in the sense

$$g^{(p)}_{klmn}\xi_{kl}\xi_{mn}\geq 0 \qquad \forall\xi\in\mathscr{S} \tag{23}$$

In a heterogeneous medium the tensor-valued function $G_{klmn}$, the measures $\mu_p$ and the tensors $G^{(p)}$ can additionally depend on **x**. The theorem presented below remains valid in the heterogeneous case but for the sake of notational simplicity we shall skip the dependence on **x** in the model parameters.

We recall that a function $f:\mathbb{R}_+\rightarrow\mathbb{R}$ (or $f:\overline{\mathbb{R}_+}\rightarrow\mathbb{R}$) is said to be completely monotonic if it has the derivatives of arbitrary orders and $(-1)^n f^{(n)}(t)\geq 0$ for $n=0,1,\ldots$ and $t>0$ (or $t\geq 0$) (Widder, 1971). The Laplace transform of the relaxation function of the anisotropic Cole–Cole model $s\rightarrow(1/s)\ \Lambda_p(s)$ is analytic in $\mathbb{C}\backslash\overline{\mathbb{R}_-}$, real and positive for $s>0$. Furthermore, it tends to 0 for $s\rightarrow\infty$. In the upper-half complex plane Im $s>0$

$$\mathrm{Im}\frac{1+a_p(s\tau_p)^{-\alpha_p}}{1+(s\tau_p)^{-\alpha_p}}=(1-a_p)\sin(\alpha_p\phi)\times\frac{(R\tau_p)^{-\alpha_p}}{|1+(s\tau_p)^{-\alpha_p}|^2}\geq 0$$

where $s=R\exp(i\phi)$. Consequently, by Theorem 5.2.6 in Gripenberg et al. (1990),

$$(1/s)\Lambda_p(s)=\int_0^{\infty}e^{-st}r_p(t)\mathrm{d}t$$

where $\lambda_p$ is locally integrable and completely monotonic. By Bernstein's theorem (op. cit. or Widder, 1971)

$$r_p(t)=\int_0^{\infty}e^{-\sigma t}\mathrm{d}\mu_p(\sigma), \qquad p=1,\ldots,P$$

which implies Eqs. (21) and (22) with $G^{(p)}=E^{(p)}\otimes E^{(p)}$.

The relaxation functions $r_p$ for the relaxation models (12)–(13)

$$a_p-\sum_{n=1}^{N}b_p^{(n)}\tau_p^{(n)}e^{-t/\tau_p^{(n)}}, \qquad t\geq 0 \tag{24}$$

$$a_p t^{-\alpha_p}/\Gamma(1-\alpha_p), \qquad t>0 \tag{25}$$

with $a_p>\Sigma_{n=1}^{N}b_p^{(n)}$, $b_p^{(n)}>0$ are obviously completely monotonic on the indicated domains of definition and Eq. (21) holds for these models.

We shall now use Eq. (21) to prove well-posedness of the initial-boundary value problem.

For this purpose we define an auxiliary field

$$\Phi_{kl}(t,\boldsymbol{x},\zeta):=\int_{-\infty}^{t}e^{-\zeta(t-\tau)}v_{k,l}(\tau,\boldsymbol{x})\mathrm{d}\tau \tag{26}$$

Obviously

$$\sigma_{kl}(t,\boldsymbol{x})=\int_0^{\infty}\Phi_{mn}(t,\boldsymbol{x},\zeta)\mathrm{d}F_{klmn}(\zeta)$$

Let $\psi_{kl}(t,\boldsymbol{x},\zeta):=\zeta\Phi_{kl}(t,\boldsymbol{x},\zeta)-[v_{k,l}(t,\boldsymbol{x})+v_{l,k}(t,\boldsymbol{x})]/2$.

The equations of motion can be recast in the following form

$$\rho\dot{v}_k(t,\boldsymbol{x})=-\partial\left\{\int_0^{\infty}\Phi_{mn}(t,\boldsymbol{x},\zeta)\mathrm{d}F_{klmn}(\zeta)\right\}\Big/\partial x_l \tag{27}$$

$$\dot{\Phi}_{kl}(t,\boldsymbol{x},\zeta)=-\Psi_{kl}(t,\boldsymbol{x},\zeta), \qquad t,\zeta\geq 0$$

$$v_k(0,\boldsymbol{x})=v_k^{(0)}(0,\boldsymbol{x}); \quad \Phi_{kl}(0,\boldsymbol{x},\zeta)=\Phi_{kl}^{(0)}(\boldsymbol{x},\zeta), \qquad \zeta\geq 0$$

where $\Phi_{kl}^{(0)}(\boldsymbol{x},\zeta):=\int_{-\infty}^{0}e^{\zeta t}[v_{k,l}^{(0)}(t,\boldsymbol{x})+v_{l,k}^{(0)}(t,\boldsymbol{x})]\mathrm{d}t/2$. Eq. (27) is equivalent to the definition of $\Phi_{kl}$, Eq. (26).

Let $\psi_{kl}^{(0)}(\boldsymbol{x},\ \zeta):=\zeta\Phi_{kl}^{(0)}(\boldsymbol{x},\ \zeta)-[v_{k,l}(0,\ \boldsymbol{x})+v_{l,k}(0,\ \boldsymbol{x})]/2$.

**Theorem 1.** *If G satisfies Eq. (21), the initial history* $v_k^{(0)}(t, \boldsymbol{x})$, $t \le 0$, *satisfies the boundary condition on* $\Gamma_1$, $\Phi_{kl}^{(0)}$ *is square integrable and*

$$\int_{-\infty}^{0} G_{klmn}(-\tau) v_{m,n}^{(0)}(\tau, \boldsymbol{x}) \mathrm{d}\tau \text{ is differentiable} \tag{28}$$

*then Eq. (27) has a unique solution* $v_k, \Phi_{kl}$ *such that* $v_k(0, x) = v_k^{(0)}(0, x)$, $k = 1, 2, 3$.

(Hanyga, 2003c).

It is worth noting that monotonicity and convexity of the stress relaxation functions $r_p$ is crucial for existence and uniqueness. It is not clear whether complete monotonicity is a necessary condition.

## 6. Regularity

Regularity results can be obtained by the semigroup techniques. They will allow us to distinguish between various models of relaxation on the basis of the ensuing wavefield properties.

The first regularity results can be obtained from Theorem 4.3 in Desch and Grimmer (1989a). Let

$$U(\beta) := (i\beta/2)[\tilde{G}(i\beta) - \overline{\tilde{G}(i\beta)}] \equiv \sum_{p=1}^{6} (1/2) \mathrm{Re} \frac{1 + a_p (i\beta\tau_p)^{\alpha_p}}{1 + (i\beta\tau_p)^{\alpha_p}} E^{(p)} \otimes E^{(p)}$$

$$V(\beta) := (\beta/2)[\tilde{G}(i\beta) + \overline{\tilde{G}(i\beta)}] \equiv \sum_{p=1}^{6} (1/2) \mathrm{Im} \frac{1 + a_p (i\beta\tau_p)^{\alpha_p}}{1 + (i\beta\tau_p)^{\alpha_p}} E^{(p)} \otimes E^{(p)}$$

Smoothness of the solutions depends on the asymptotic behavior for large $\beta > 0$ of a function $\gamma(\beta)$ defined by the inequality $\beta V(\beta) - \gamma(\beta) U(\beta) \ge 0$. The left-hand side of this inequality is interpreted as a symmetric operator on $\mathscr{S}$. Since

$$\beta \frac{\mathrm{Im}\{[1 + a_p (i\tau_p\beta)^{-\alpha_p}][1 + (-i\tau_p\beta)^{-\alpha_p}]\}}{\mathrm{Re}\{[1 + a_p (i\tau_p\beta)^{-\alpha_p}][1 + (-i\tau_p\beta)^{-\alpha_p}]\}} \ge \frac{\min\{\tau_p^{-\alpha_p}(1 - a_p)\sin(\pi\alpha_p/2)\beta^{1-\delta(\beta)}\}}{[1 + \max\{a_p\tau_p^{-\alpha_p}\}\beta^{-\varepsilon(\beta)}][1 + \max\{\tau_p^{-\alpha_p}\}\beta^{-\varepsilon(\beta)}]}$$

where

$$\delta(\beta) = \begin{cases} \min\{\alpha_p\} & \text{for } \beta < 1 \\ \max\{\alpha_p\} & \text{for } \beta > 1 \end{cases}$$

and

$$\varepsilon(\beta) = \begin{cases} \max\{\alpha_p\} & \text{for } \beta < 1 \\ \min\{\alpha_p\} & \text{for } \beta > 1 \end{cases}$$

$\gamma(\beta) \sim \text{const} \times \beta^{1-\delta(\beta)}$ for $\beta \to \infty$ and Theorem 4.3 in Desch and Grimmer (1989a) implies that the solution of Eq. (27) belongs to $\mathscr{C}^\infty(\mathbb{R}_+, H^1(\Omega, \mathbb{R}^3))$. The same criterion also implies finite speed of propagation of disturbances (Prüss, 1993).

For the fractional relaxation model $\beta/\gamma(\beta) = \max\{\cot(\pi\alpha_p/2) | p = 1, \ldots, P\}$ provided $0 < \alpha_p < 1$, and the solution is an analytic function of time. For $\alpha_p > 0$ the kernel $G'$ relating the delayed stress to the strain has a nonintegrable singularity at 0. The problem is parabolic and the propagation speed of disturbances is infinite. The speed of signal propagation can, however, be finite. For example, if $\alpha_p = \alpha$ does not depend on $p$ then the Green's function depends on $t$, $\boldsymbol{x}$ through the variable $\xi = x/t^{(1-\alpha)/2}$ (Hanyga, 2002b; Carcione et al., 2002). For $\alpha_p < 0$ the well-posedness fails because the stress relaxation functions $r_p$ are increasing.

## 7. Numerical methods for the anisotropic Cole–Cole model

In engineering FEM methods in combination with discretized fractional derivatives have been commonly used in modeling of isotropic viscoelastic structures with fractional relaxation (Padovan, 1987), in particular for the Cole–Cole relaxation (Schmidt et al., 2000).

For $\tau_p$ independent of $p = 1, \ldots, 6$ and a homogeneous medium, the anisotropic Cole–Cole equations can be formulated in terms of fractional derivatives. The fractional derivative formulation immediately provides several alternative time discretizations. More general time-stepping algorithms are described in Hanyga (2002a). Alternatively, wave propagation can be expressed in terms of a coupled system consisting of the partial differential equations expressing the usual equations of motion and additional ordinary differential equations of fractional order which may be interpreted as relaxation laws (Hanyga, 2003b).

For rational values of $\alpha_p$, ray-asymptotic methods (based on real ray tracing) can be applied following the methods of Hanyga and Seredyńska (1999a). This is due to the reducibility of the problem to the tracing of a wavefront in real space while the signal shape is represented by convolution with a set of functions representing the evolution of a delta signal. These functions are explicitly calculable for $\alpha_p = 1/2, 1/3, 2/3$ (Hanyga and Seredyńska, 2002). For $\alpha_p = 1/2$, ray tracing requires integration of a single additional transport equation in order to determine the degree of signal spreading and the delay of the signal peak with respect to the wavefront. In contrast, ray-asymptotic methods for the general case of a viscoelastic medium require the use of complex ray tracing in order to capture dispersion and attenuation (Hanyga, 1999a; Hanyga and Seredyńska, 2000).

Essentially identical existence, uniqueness and regularity results have been obtained for linear thermoviscoelasticity with heat conduction (Hanyga, 2003c). In particular, regularity properties depend on the memory kernel and are not affected by thermal effects.

## 8. Conclusions

Well-posedness has been proved for a class of viscoelastic models constructed by the method of Carcione and Cavallini (1994) with stress relaxation functions enjoying appropriate monotonicity and convexity properties. In this framework the three models of viscoelastic relaxation currently enjoying some popularity in seismic modeling, viz. exponential relaxation (Eq. (24)), fractional relaxation (Eq. (25)) and Cole–Cole relaxation (Eq. (4)), lead to well-posed problems. The first model lacks the smoothing property characteristic of viscosity, the second one is characterized by an infinite speed of propagation of disturbances while the third combines a finite propagation speed with a smoothing effect of Newtonian viscosity.

In an anisotropic medium the constant $Q$ property cannot be imposed by a constitutive assumption (Hanyga, 2001a) unless all the six relaxation functions are associated with the same $Q$ value (the isotropic case is an exception). The propagation speed is unbounded and perturbations spread immediately to the entire domain while a finite signal speed, resulting from self-similarity, is an attribute of the scalar equation with constant coefficients. It need not be valid for the anisotropic case. On the other hand, the relative advantage of the Cole–Cole model—good matching of experimental data with an economic set of parameters—remains true in the anisotropic case.

It is worth noting that the Cole–Cole law and related viscoelastic models can be expressed in terms of 2–6 internal variables subject to fractional relaxation laws, hence the integral operator representing the memory can be replaced by a set of at most six ordinary differential equations of fractional order (Hanyga, 2003b). For numerical implementation this implies that a generalization of the method developed by Day and Minster (1984), Emmerich and Korn (1987), Simo and Hughes (1998) and Carcione et al. (1988) for exponential relaxation can be applied. This approach can be extended to nonlinear viscoelastic models of rate-dependent hysteresis (Hanyga, 2003b).

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