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GEOPHYSICS

Nonlinear Dynamics of Slope Flows: Simple Models and Exact Solutions

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Regular observations at the Northern Caucasus Geophysical Observatory recorded on March 10, 2006, a catastrophic avalanche, which caused disaster and fatalities. The instruments located immediately beneath the avalanche development zone in the tunnel of the Baksan neutrino observatory recorded detailed structure of the seismic and magnetic fields at the stage of preparation and descent of the avalanche. We obtained unique data that make it possible to correlate them with independently measured characteristics of the avalanche (volume of the snow mass, velocity of motion, moments of contacts with uneven features of the pathway, and so on). However, simple models and formulas for the specific analysis are lacking. Therefore, we propose a theoretical basis for comparison with the field experiment data.

In order to get a clear understanding of the nonlinear dynamics of avalanches [1, 2], it is useful to have exact solutions of the problems even in simplified formulation. Such solutions are also needed for testing the programs for the numerical modeling of more complex problems.

Let an incompressible medium ($\rho = \rho_0$) move down the slope with angle α to the horizon in the gravity force field $\rho_0 g$. The x-axis is oriented down the slope; the y-axis is oriented normal to the slope in the upward direction. We assume that the velocity changes along the slope more slowly than across the slope and that the longitudinal velocity component u is much greater than the transverse component v. The approximations of such type are used in hydrodynamics (theory of boundary layer) and in wave physics (quasi-optical approximation of diffraction theory) [3].

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Keeping the terms of the order of smallness not greater than the first, we get the following relations for pressure p and elevation of the free surface $y = h(x, t)$ from the projection of the equation of motion on the y-axis

$$\rho_0^{-1}\frac{\partial p}{\partial y} + g\cos\alpha = 0, \quad p = g\rho_0\cos\alpha[h(x,t) - y]. \quad (1)$$

Substituting (1) into the projection of the equation on the x-axis, we come to the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\cos\alpha\frac{\partial h}{\partial x} + g\sin\alpha. \quad (2)$$

We add the continuity equation and kinematic relation

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = v. \quad (3)$$

At the surface, $y = h(x, t)$. It is clear that $v = 0$ at the rigid slope.

We seek solution of (2), (3) as $u = U(x, t)$, $v = -y\frac{\partial U}{\partial x}$, where $U(x, t)$ is an unknown function. The continuity equation and the boundary condition at the slope are satisfied identically, while Eq. (2) and boundary condition (3) form the following system

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\cos\alpha\frac{\partial h}{\partial x} = g\sin\alpha,$$
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hU) = 0. \quad (4)$$

System (4) has an exact solution. In order to find it, we assume

$$U = tg\sin\alpha + 2\sqrt{g\cos\alpha \cdot h(x,t)}. \quad (5)$$

Substitution of (5) reduces each of Eqs. (4) to one equation

$$\frac{\partial h}{\partial t} + (tg\sin\alpha + 3\sqrt{g\cos\alpha}\sqrt{h})\frac{\partial h}{\partial x} = 0. \quad (6)$$

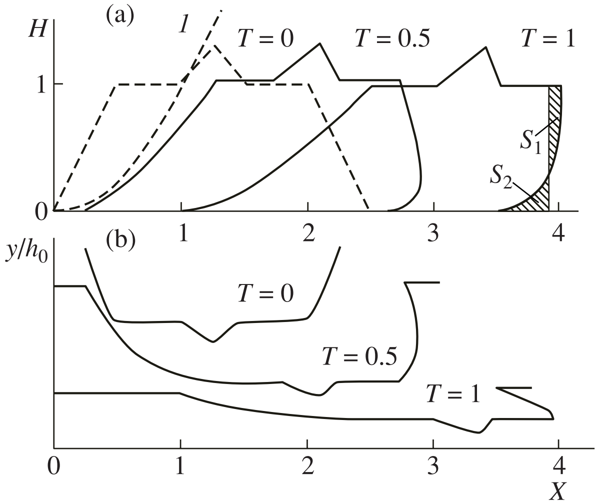

Fig. 1. (a) Form of the flow surface at different time moments. Dashed curve is the initial form. Solid curves show the surface at $T = 0.5$ and $T = 1$. Curve *1* is the parabola $H = \left(\frac{X}{aT}\right)^2$. (b) Form of the streamlines for the shown profiles.

The exact solution corresponding to the specified form of the initial surface of the medium over slope $h = h_0\Phi\left(\frac{x}{x_0}\right)$ is an implicit function

$$h(x, t) = h_0\Phi(x_0^{-1}(x - 0.5gt^2\sin\alpha - 3t\sqrt{g\cos\alpha}\sqrt{h(x, t)})). \quad (7)$$

We introduce the following dimensionless notations for convenience

$$\frac{h}{h_0} = H,\ \frac{x}{x_0} = X,\ t\sqrt{\frac{g\sin\alpha}{2x_0}} = T,\ a = 3\sqrt{\frac{2h_0}{x_0\tan\alpha}} \quad (8)$$

and write the solution of (7) as the explicit function $X(T, H)$:

$$X = \Phi^{-1}(H) + aT\sqrt{H} + T^2. \quad (9)$$

We shall analyze the distortions of the surface using the graphic method described in [4].

As seen from relation (9), the inverse profile of $X(H)$ is obtained by adding the initial profile $X = \Phi^{-1}(H)$ (where $\Phi^{-1}(H)$ is the inverse function to $\Phi(X)$) and curve $X = aT\sqrt{H}$. Then, the profile should be shifted along X over distance T^2.

The profile of the flow is shown in Fig. 1 for time moments $T = 0$ (dashed line), $T = 0.5$, and $T = 1$ (solid curves). The parameter is $a = 1$.

The streamlines calculated from the formula

$$\frac{y}{h_0} = \frac{C_1}{U(x, t)} = \frac{C}{T + \frac{a}{3}\sqrt{H(x, T)}} \quad (10)$$

are shown in Fig. 1 (here, C_1 and C are constants).

The ambiguity of function $H(X)$ at the leading front of the flow (Fig. 1a) should provoke a breaking of its overhanging part. It can be eliminated by introducing a discontinuity on the basis of equal squares $S_1 = S_2$ [5] (Fig. 1a):

$$\frac{d}{dT}\int_{H_1}^{H_2}[X(H) - X_{SH}(T)]dH = 0. \quad (11)$$

Here, $X_{SH}(T)$ is the discontinuity coordinate (see Fig. 1a); H_1, H_2 are the values of the variable immediately before and after the discontinuity. Substituting solution (9) into (11), we find the front propagation velocity. In dimension variables, it is equal to

$$\frac{dx_{SH}}{dt} = gt\sin\alpha + 2\sqrt{h_0 g\cos\alpha}\frac{h_1 + \sqrt{h_1h_2} + h_2}{\sqrt{h_1} + \sqrt{h_2}}. \quad (12)$$

The first term in (12) (which increases with time) is related to the downslope force; the second term is a nonlinear additive to the velocity.

The velocity increase should be limited by losses due to friction between the flow and the slope. We shall modify system (4) to take into account the slowdown [2]

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\partial \zeta}{\partial x} = b - \varphi(U),$$
$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(\zeta U) = 0. \quad (13)$$

Here, $\zeta = g\cos\alpha \cdot h(x, t)$, $b = g\sin\alpha$, and $\varphi(U) = \beta U + kU^2$. The term with coefficient β describes the losses of the type of viscid friction, and the term with k describes nonlinear resistance. If the thickness of the downslope flow varies smoothly and we can neglect derivative $\frac{\partial \zeta}{\partial x}$ in the first equation of (13), Eq. (13) is reduced to a system of quasi-linear equations of the first order with an exact solution. These equations have a common main part [6]. The solution is cumbersome. For $\beta = 0$,

$$kx + \ln\mathrm{ch}(kDt) + \ln\left[1 - \frac{U}{D}\tanh(kDt)\right] = kx_0\Phi_U^{-1}\left[\frac{D}{D_0}\frac{U - D\tanh(kDt)}{D - U\tanh(kDt)}\right], \quad (14)$$

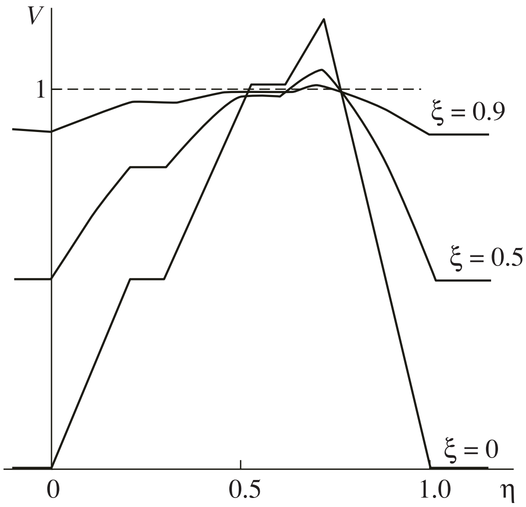

Fig. 2. Transformation of the flow velocity profile with increasing time variable ξ. The auxiliary spatial coordinate (parameter η in Eq. (16)) is shown along the abscissa axis.

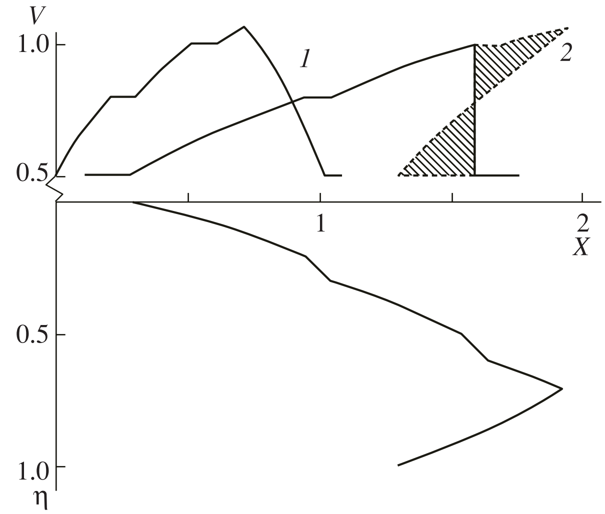

Fig. 3. Transformation of the flow velocity profile as a result of nonlinear deformation of the abscissa axis for the profile corresponding to moment ξ = 0.5.

$$\frac{\partial\zeta}{\partial x} = \left(\frac{\partial\zeta}{\partial x}\right)_0 \Phi_{\zeta'}\left[\frac{x}{x_0} + \frac{1}{kx_0}\ln\left\{\cosh(kDt)\left[1 - \frac{U}{D}\tanh(kDt)\right]\right\}\right] \quad (15)$$

Here, $D^2 = \frac{g}{k}\sin\alpha$; Φ_U and $\Phi_{\zeta'}$ are arbitrary functions. This solution can be analyzed more easily if we write it in dimensionless parametric form:

$$V = \frac{\xi + V_0\Phi_U\left(\frac{\eta}{X_0}\right)}{1 + \xi V_0\Phi_U\left(\frac{\eta}{X_0}\right)}, \quad \frac{\partial\zeta}{\partial x} = \left(\frac{\partial\zeta}{\partial x}\right)_0 \Phi_{\zeta'}\left(\frac{\eta}{X_0}\right), \quad (16)$$

$$\eta = X + \ln(1 - \xi^2) - \ln(1 - V\xi)^2.$$

Here, $\xi = \tanh(kDt)$, $V = \frac{U}{D}$, $V_0 = \frac{U_0}{D}$, $X = 2kx$, $X_0 = 2kx_0$, η is a parameter coinciding with coordinate X at the initial time moment.

The scheme of the graphic solution of (16) is as follows. First, we construct profile $V\left(\frac{\eta}{X_0}\right)$ at a given profile $\Phi_U\left(\frac{\eta}{X_0}\right)$ and increasing the temporal variable $\xi = \tanh(kDt)$, $0 < \xi < 1$. Then, we construct the coordinate as a function of the parameter:

$$X(\eta) = \eta + \ln\left[1 + \xi V_0\Phi_U\left(\frac{\eta}{X_0}\right)\right]^2 - \ln(1 - \xi^2). \quad (17)$$

Next, we find a reverse function for $\eta(X)$. Finally, deforming the abscissa axis of the initial profile, we find profile $V[\eta(X)]$ at the current time moment.

The scheme described above was realized in Figs. 2 and 3 for $X_0 = 1$. First, profiles $V(\eta)$ for $\xi = 0$, 0.5, and 0.9 were plotted in Fig. 2. The value $\xi = 0$ corresponds to the initial profile of arbitrary form. It is seen that the indented profile becomes smoother with time and small details disappear. The values of velocity $V(\eta)$ exceeding unity decrease with time, while the values smaller than unity increase. In general, the velocity profile becomes more uniform. At any point η, the value of $U \to D = \sqrt{\frac{g}{k}\sin\alpha}$ at $\xi = \tanh(kDt) \to 1$.

In order to take nonlinear distortions into account, it is necessary to deform the abscissa axis according to (17) using the data in Fig. 2. The dependence of $X(\eta)$ for $\xi = 0.5$ is shown in the lower part of Fig. 3. It is not monotonic, which leads to ambiguity of $V(X)$ (curve 2) at such large times. The formation of the flow front (an interval that cuts off dashed areas) smoothes even more the spatial details of the profile and stretches notably the region of the flow in which $V > 0.5$. Curve *1* coincides with curve $V(\eta)$ at time moment $\xi = 0.5$ in Fig. 2.

Let us now consider the propagation of finite perturbations along the surface of a uniform flow (not depending on the coordinate) (see [7]). We specify in (13) $U = U_0(t) + U'(x, t)$, $\zeta = \zeta_0 + \zeta'(x, t)$. As follows from (13), velocity U_0 satisfies the equation

$$\frac{dU_0}{dt} = b - \beta U_0 - kU_0^2. \quad (18)$$

The solution of (18) with zero initial condition is written as

$$U_0(t) = \frac{\beta}{2k}\left[\frac{1+\frac{\gamma}{\beta}\tanh\frac{t\gamma}{2}}{1+\frac{\beta}{\gamma}\tanh\frac{t\gamma}{2}} - 1\right], \quad \gamma = \sqrt{4bk+\beta^2} \quad (19)$$

It is seen that the velocity of a uniform flow increases from zero to a stationary value

$$U_\infty = \frac{\beta}{2k}\left[\frac{\gamma}{\beta} - 1\right], \quad (20)$$

which is gained at $t \to \infty$. In two limiting cases, when either viscous friction or nonlinear resistance dominates, the steady velocity of a uniform flow becomes equal to $U_\infty = \frac{b}{\beta}$ or $U_\infty = \sqrt{\frac{b}{k}}$, respectively. Substituting (19) into equation system (13) for velocity perturbations and elevation of the surface, we get a pair of equations

$$\frac{\partial U'}{\partial t} + \frac{\partial \zeta'}{\partial s} = -U'\frac{\partial U'}{\partial s} - \beta U' - 2kU_0(t)U' - kU'^2,$$
$$\frac{\partial \zeta'}{\partial t} + \zeta_0\frac{\partial U'}{\partial s} = -\frac{\partial}{\partial s}(\zeta' U'). \quad (21)$$

Here, we introduced a new coordinate $s = x - \int_0^t U_0(t')dt'$.

Small linear and nonlinear terms are gathered in the right-hand parts of (21). From hereon, we shall omit the primes in variables describing the perturbation.

We shall use the method of a slowly varying profile [3–5] for simplifying nonlinear system (21). We seek the solution in the form of a wave running down the slope with a slow variation of its shape: $\zeta = \zeta(t_1 = \mu t, S = s - t\sqrt{\zeta_0})$. As a result of a standard procedure [4], we reduce system (21) to the equation

$$\frac{\partial \zeta}{\partial t} + \frac{3}{2\sqrt{\zeta_0}}\zeta\frac{\partial \zeta}{\partial S} = -\left[\frac{\beta}{2} + kU_0(t)\right]\zeta - \frac{k}{2\sqrt{\zeta_0}}\zeta^2. \quad (22)$$

According to the solution of (19), the expression in square brackets in (22) is transformed as follows

$$\frac{\beta}{2} + kU_0(t) = \frac{\gamma}{2}\tanh\left(\frac{t+t_0}{2}\gamma\right), \quad (23)$$

where t_0 is determined by relation $\tanh\frac{\gamma t_0}{2} = \frac{\beta}{\gamma}$, $\gamma \equiv \sqrt{4bk+\beta^2}$. Introducing new variables

$$G = \zeta\cosh\left(\frac{\gamma}{2}(t+t_0)\right),$$
$$q = \frac{1}{\gamma\sqrt{\zeta_0}}\left[\arctan\sinh\left(\frac{\gamma}{2}(t+t_0)\right) - \arctan\sinh\left(\frac{\gamma}{2}t_0\right)\right], \quad (24)$$

we transform the equation to the simplest form

$$\frac{\partial G}{\partial q} + 3G\frac{\partial G}{\partial S} = -kG^2. \quad (25)$$

The exact solution to (25) is given by the implicit function

$$G = (1-kqG)\Phi\left[S + \frac{3}{k}\ln(1-kqG)\right]. \quad (26)$$

At greater times, decay decreases the proportion of nonlinear term $3G\frac{\partial G}{\partial S}$ in Eq. (25) to the distortion of the profile. The wave velocity is determined from the condition of the constant phase

$$S = x - t\sqrt{\zeta_0} - \int_0^t U_0(t')dt' = \text{const}. \quad (27)$$

Substituting $U_0 \to U_\infty$ in (27) at large times, we find the steady wave velocity without taking into account nonlinear phase effects:

$$C_{LIN}(t\to\infty) = \sqrt{gh_0\cos\alpha} + \frac{\sqrt{4gk\sin\alpha + \beta^2} - \beta}{2k}. \quad (28)$$

At small values of reduced time q during the acceleration of the flow, one can use approximate solution $G = \Phi[S - 3qG]$ instead of (27). In this case, the velocity of wave propagation along the flow is equal to

$$C(t\to 0) \approx \sqrt{gh_0\cos\alpha} + gt\sin\alpha + \frac{3h'\sqrt{g\cos\alpha}}{2\sqrt{h_0}}. \quad (29)$$

It is necessary to scrutinize solution (26) to describe the dependence of velocity on time and perturbation. It is important to take into account the roughness of the slope, as was done for sea waves in [8]. These clarifications would be done later during processing of the experimental data.

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REFERENCES

1. A. N. Bozhinskii and K. S. Losev, *Fundamentals of Avalanche Science* (Gidrometeoizdat, Leningrad, 1987) [in Russian].

2. S. S. Grigoryan and A. V. Ostroumov, *A Mathematical Model of Slope Processes of Avalanche Type*: *Report no. 1375 of the Research Institute of Mechanics, Moscow State University* (MGU, Moscow, 1977) [in Russian].

3. M. B. Vinogradova, O. V. Rudenko, and A. P. Sukhorukov, *Theory of Waves* (Nauka, Moscow, 1990) [in Russian].

4. O. V. Rudenko and S. I. Soluyan, *Theoretical Principles of Nonlinear Acoustics* (Nauka, Moscow, 1975) [in Russian].

5. O. V. Rudenko, S. N. Gurbatov, and K. M. Khedberg, *Nonlinear Acoustics in Examples and Problems* (Fizmatlit, Moscow, 2006) [in Russian].

6. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, (Interscience Publishers, New York 1953; Gostekhteorizdat, Leningrad, 1945), Vol. 2 [in Russian].

7. V. E. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, *Wave Dynamics of Gas and Vapor Fluid Systems* (Energoatomizdat, Moscow, 1990) [in Russian].

8. L. A. Ostrovskii and E. N. Pelinovskii, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **6**, 934 (1970).