
Analysis of radial movement of an unconfined leaky aquifer due to well pumping and injection

Jiang Li

Abstract Radial movement of an unconfined leaky aquifer was studied with respect to hydraulic forces that are induced by well recharge and discharge. New analytic solutions in the velocity and displacement fields were found and applied to describe transient movement in an unconfined leaky aquifer. Linear momentum and mass balance of saturated porous sediments, the Darcy-Gersevanov law, and the analytic solution of hydraulic drawdown for unsteady flow within the unconfined leaky aquifer were introduced to find the new solutions. Analytic results indicate that the nonlinear relation between the initial hydraulic head (h_0) and the well function has an insignificant effect on the aquifer transient movement when the drawdown $s < 0.02h_0$. When the well function is simplified with different assumptions and pumping conditions, the new solutions correspondingly reduce to cases that are similar to the Hantush-Jacob, Muskat, and Theis transient movement of a confined leaky aquifer. It was found that large leakance is important in slowing radial movement and reducing aquifer deformation. Flow velocity in the aquifer is more responsive to leakance than to cumulative displacement within the aquifer. The zones and boundary with tensile stress can be located using the same approach applied to a confined aquifer for risk assessment of earth fissuring.

Résumé Le mouvement radial dans un aquifère semi-perméable libre a été étudié en considérant les forces hydrauliques induites par la recharge et le prélèvement au niveau d'un puits. De nouvelles solutions analytiques des champs de vitesse et de déplacement ont été trouvées et appliquées pour décrire le mouvement transitoire dans un aquifère semi-perméable libre. Pour trouver les nouvelles solutions, le moment linéaire et le bilan de masse dans les sédiments poreux saturés, la loi de Darcy-Gersevanov, et la solution analytique du rabattement hydraulique pour un écoulement transitoire dans un aquifère semi-perméable

libre sont pris en compte. Les résultats analytiques montrent que la relation non-linéaire entre le potentiel hydraulique initial (h_0) et la fonction de puits a un effet insignifiant sur le mouvement transitoire de l'aquifère quand le rabattements $s < 0.02h_0$. Si la fonction de puits est simplifiée en tenant compte de différentes hypothèses et conditions de pompage, les nouvelles solutions se réduisent en conséquence à des cas similaires au mouvement transitoire dans un aquifère semi-perméable captif selon Hantush-Jacob, Muskat, et Theis. Il s'est avéré qu'une drainance élevée est importante pour le ralentissement du mouvement radial et la réduction de la déformation de l'aquifère. La vitesse d'écoulement dans l'aquifère est plus sensible à la drainance qu'au déplacement cumulé dans l'aquifère. Les zones et la frontière présentant des contraintes de traction peuvent être localisées en utilisant la même approche appliquée à un aquifère captif pour l'évaluation des risques de fissuration du sol.

Resumen Se ha estudiado el movimiento radial en un acuífero no confinado con fuga en relación a las fuerzas hidráulicas que son inducidas por la descarga y recarga del pozo. Se encontraron nuevas soluciones analíticas en los campos de desplazamiento y velocidad, las cuales se aplicaron para describir el movimiento transitorio en un acuífero no confinado con fuga. Se han introducido cálculos de balance de masa y momentum lineal de los sedimentos porosos saturados, la Ley Darcy-Gersevanov, y la solución analítica de descenso hidráulico para flujo sin régimen permanente dentro del acuífero no confinado con goteo para encontrar las nuevas soluciones. Los resultados analíticos indican que la relación no lineal entre la presión hidráulica inicial (h_0) y la función del pozo tiene efecto insignificante en el movimiento transitorio del acuífero cuando el descenso $s < 0.02h_0$. Cuando se simplifica la función del pozo en base a diferentes supuestos y condiciones de bombeo se reducen las nuevas soluciones a casos que son similares a el movimiento transitorio Hantush-Jacob, Muskat, y Theis de un acuífero confinado con goteo. Se encontró que las fugas grandes son importantes en la disminución del movimiento radial y en la reducción de la deformación del acuífero. La velocidad de flujo en el acuífero responde más a las fugas que al desplazamiento acumulativo dentro del acuífero. Pueden localizarse zonas y límites con esfuerzos tensionales usando el mismo enfoque aplicado a acuíferos

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J. Li (✉)
Morgan State University,
5200 Perring Parkway,
Baltimore, MD 21251-0001, USA
e-mail: jli@eng.morgan.edu

confinados para evaluación de riesgo de generación de fracturas en el terreno.

Keywords Aquifer transient movement · Unconfined leaky aquifer · Aquifer storage and recovery · Analytic solutions · Groundwater flow

Nomenclature

b	Thickness of a confined aquifer, L
B	$(\equiv r/2\lambda)$ the dimensionless leaky factor
D_0	$(\equiv 2\pi br)$ a parameter, L^2
D_1	$(\equiv S_s r^2/4K)$ a parameter, T
g	Gravitational acceleration, L/T^2
h	Hydraulic head in the pumped unconfined aquifer, L
h_c	Hydraulic head in the unpumped confined aquifer, L
h_0	Initial hydraulic head, L
h^*	$(\equiv h/h_0)$ the dimensionless hydraulic head, L
k_0	a constant
K	Radial hydraulic conductivity, L/T
$\mathbf{K}$	Hydraulic conductivity tensor, L/T
m	A dummy variable of integration
m_{gen}	Cumulative and generated mass, M
m_j	$(j=b, w, s)$ Mass of bulk, water and soil, M
n	Porosity, dimensionless
q	Radial specific discharge, L/T
$\mathbf{q}$	$[\equiv n(\mathbf{v}_w - \mathbf{v}_s)]$ specific discharge, L/T
Q_j	$(j = w, \text{leak})$ Flow rate by volume for water and leakage, L^3/T
$Q(t)$	Well pumping-injection rate by volume, L^3/T
r	Radial distance from the center of pumping well, L
r_w	Radius of the pumping well, L
r^*	$[\equiv r(S'/T_0 t)^{1/2}]$ dimensionless radial distance of r
r^*_{max}	Dimensionless radial distance for the maximum radial displacement
S	Specific yield of an unconfined aquifer, $1/L$
S'	Apparent storage coefficient
t	Time, T
t^*	$(\equiv tT_0/S'r^2)$ dimensionless time
T_0	$(\equiv h_0 K)$ initial transmissivity, L^2/T
T_{avg}	$(\equiv h_{\text{avg}} K)$ average transmissivity, L^2/T
u_{max}	Maximum value of u_j , L
$u_{s,u}$	Displacement of solid, L
u^*	Dimensionless radial displacement for u_j
u^*_{max}	Maximum value of u^*
v_i	$(i=b, w, s)$ Radial velocities of bulk, water, and solid flow, L/T
$\mathbf{v}_i$	$(i=b, w, s)$ Velocities of water and solid flow, L/T
v_{max}	Maximum value of v_s , L/T
v^*_{max}	Maximum value of v^*
α	Angle between flow and the normal vec-

ϵ^*	tor on the surface $(\equiv \partial u^*/\partial r)$ strain based on the dimensionless displacement
ξ	$(\equiv S'r^2/tT_0)$ a dimensionless variable
θ	Tangential coordinates in an axis-symmetric system
λ	$[\equiv (b'T_0/K)^{1/2}]$ the leaky factor, L
ρ_i	$(i=b, w, s)$ Densities of bulk, water and solid flow, M/L^3
∇	Differential operator for gradient
$\partial/\partial r$	Partial derivative with respect to r , $1/L$
$\partial/\partial t$	Partial derivative with respect to time t , $1/T$
$\equiv$	By definition
Subscripts	
0	Initial values of variables at $t=0$
gen	Generated mass
b	Bulk flow
gen	Generated mass
max	Maximum values of velocity and displacement
s	Solid
w	Water
Superscripts	
*	Dimensionless variables
'	Variables related to the confining bed

Introduction

The technology of aquifer storage and recovery (ARS) in aquifer systems is widely applied in the United States for the management of water resources and the improvement of water quality. Currently, over 59 ASR operating sites are located in more than 16 States (Pyne et al. 1995). The applications of artificial discharge and recharge may raise potential risk of earth fissuring because pumping-injection activity induces hydraulic forces that may drive the aquifer matrix to move horizontally toward or away from a pumping well. The magnitude of the maximum cumulative horizontal displacement due to ground fluid withdrawal can be the same order as that of the maximum cumulative vertical displacement or subsidence (Yerks and Castle 1969). The mechanism of earth fissuring at depth generated by hydraulic forces was previously studied and discussed (Helm 1994a,b; Sheng et al. 2003; Li 2006, 2007). Traditionally, aquifer movement was investigated with emphasis on the stress and strain fields in an aquifer (Biot 1941; Terzaghi 1943; Mikasa 1965; Helm 1987; Li 2003). The approach applied in the present report is substantially different from the traditional approaches and emphasizes the flow relation in the velocity and displacement fields. Analysis of the aquifer radial movement in response to a constant pumping rate in a discharging well was first conducted by Helm (1994b) using an analytic approach incorporated with the Theis solution (Theis 1935), which requires constant pumpage. A constant well-flow rate, however, cannot be applied to the transient aquifer movement caused by the artificial discharge-recharge

activity that features a well-flow rate changing with time. Li (2006, 2007) investigated transient radial movement of confined leaky and nonleaky aquifers in response to various well-flow rates that (e.g., linear, nonlinear, periodic, and exponential) vary with time so that the ASR activity in fields can be simulated. This paper, with emphasis on an unconfined leaky aquifer, is considered as a companion investigation of Li's previous studies for confined nonleaky and leaky aquifers. The purpose of this report is to find new analytic solutions for an unconfined leaky aquifer in response to the pumping-injection activity. To analyze radial motion of the unconfined leaky aquifer, linear momentum and mass balance of bulk mass (saturated soil), and the Darcy-Gersevanov law (Gersevanov 1937) need to be introduced first.

Linear momentum of bulk mass flux

The average linear momentum for saturated bulk mass flow at a specified point P of interest located at the centroid of a representative elementary volume (REV) (Bear 1972) can be introduced as shown below:

$$m_b \mathbf{v}_b = m_w \mathbf{v}_w + m_s \mathbf{v}_s \quad (1)$$

where m_i and $\mathbf{v}_i$ ($i=w, s$, or b) are mass and average velocity of water, solid, and bulk mass flow at the point P carefully defined within the REV in Fig. 1. Subscripts w , s , and b denote water, solid, and bulk, respectively. A bold variable represents a vector (e.g., $\mathbf{v}_w$ and $\mathbf{v}_s$). If the terms on both sides of (1) are divided by the bulk mass m_b , the bulk velocity takes the form:

$$\mathbf{v}_b = \frac{m_s}{m_b} \mathbf{v}_s + \frac{m_w}{m_b} \mathbf{v}_w \quad (2)$$

or alternatively:

$$\mathbf{v}_b = \frac{\rho_s}{\rho_b} (1 - n) \mathbf{v}_s + \frac{\rho_w}{\rho_b} n \mathbf{v}_w = \mathbf{v}_s + \frac{\rho_w}{\rho_b} \mathbf{q} \quad (3a)$$

where $m_b \equiv \rho_b V_b$, $m_w \equiv \rho_w V_w$, and $m_s \equiv \rho_s V_s$ in which m_i , ρ_i and V_i ($i=b, w$ and s) are mass, density, and volume of bulk, water, and solid; $\mathbf{q} [\equiv n(\mathbf{v}_w - \mathbf{v}_s)]$ is the average specific discharge; $\rho_b [\equiv n\rho_w + (1-n)\rho_s]$ is the bulk density; n is porosity and equals V_w/V_b for saturated soil. For saturated porous sediments, bulk mass and volume can be respectively expressed by $m_b \equiv m_w + m_s$ and $V_b \equiv V_w + V_s$. The bulk velocity $\mathbf{v}_b$ in (2) is the resulting average velocities $\mathbf{v}_w$ and $\mathbf{v}_s$ weighted by fractions of constituent mass m_w/m_b and m_s/m_b that individually equal $n\rho_w/\rho_b$ and $(1-n)\rho_s/\rho_b$. The bulk velocity $\mathbf{v}_b$ in (2) is presented by a small parallelogram in dash lines in Fig. 1. At the same time, $\mathbf{v}_b$ in terms of $\mathbf{v}_s$ and $\mathbf{q}$ in (3a) is illustrated by a triangle in solid lines (Fig. 1). In an axisymmetric coordinate system, $\mathbf{v}_b$ has three components

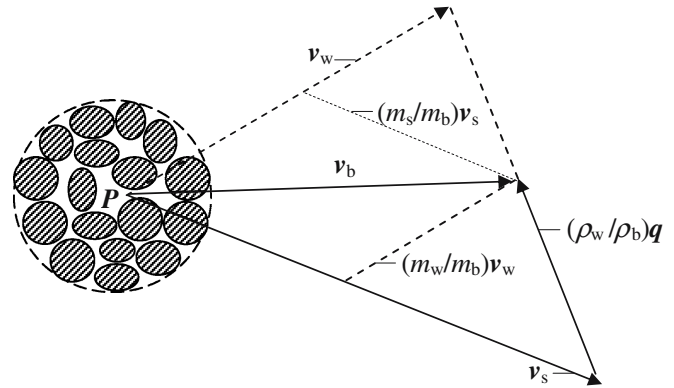


Fig. 1 The average linear momentum of the bulk mass flux at P

in vertical, radial, and tangential directions, i.e., $\mathbf{v}_b(r, \theta, z) = \{v_{br}, v_{b\theta}, v_{bz}\}$. Particularly, the radial component in (3a) equals $v_{br} = v_{sr} + (\rho_w/\rho_b)q_r$, where scalars q , v_b , and v_s are specific discharge, bulk, and solid velocities, respectively, in the radial direction and the subscript r for the radial direction is dropped for simplicity of notation in the rest of the report.

The mass bulk flow convention of $\mathbf{v}_b$ expressed above by Eqs. (2) and (3a) should not be confused with Helm's (1994b) convention of volume bulk flux $\mathbf{q}_b$. Helm's convention of the bulk volume flux in a saturated aquifer is defined as $V_b \mathbf{q}_b \equiv V_w \mathbf{v}_w + V_s \mathbf{v}_s$ that can result in:

$$\mathbf{q}_b = \mathbf{v}_s + \mathbf{q} \quad (3b)$$

The bulk mass flux $\mathbf{v}_b$ is normally invoked for conservation of linear momentum as presented in (1), whereas the bulk volume flux $\mathbf{q}_b$ is introduced for conservation of water mass released from aquifer storage. It should be emphasized that using either $\mathbf{v}_b$ or $\mathbf{q}_b$ leads to an identical interpretation of the same physical processes. Hence, basic governing equations that are based on either convention will be found to be identical or interchangeable. However, the conceptual and mathematical roads to the basic governing equations may be distinct.

Mass balance for bulk mass flux

Mass balance for a saturated and mixed bulk mass flow with two phases (soil and water) over a volume V of interest requires (Helm 1987):

$$\int_A \rho_b \mathbf{v}_b \cdot d\mathbf{A} + \int_V \frac{\partial \rho_b}{\partial t} dV = \frac{\partial m_{gen}}{\partial t}, \quad (4)$$

where t is time, A is the surface covering the volume V of interest in the aquifer, and $\partial m_{gen}/\partial t$ is the mass rate generated by sinks or sources (e.g., springs or drains) within the volume. On the left side of (4), the first term represents the bulk mass flow crossing the surface that

covers the volume V , and the second term is the net cumulative rate of mass within the volume with respect to time.

Considering a saturated cylindrical zone with the radius r from the center of a fully penetrating well in the unconfined leaky aquifer (Fig. 2) and assuming the radius r_w of the discharging well to be negligibly small result in an alternative expression of (4):

$$\int_{A_1} \rho_{b1} \mathbf{v}_{b1} \cdot d\mathbf{A}_1 + \int_{A_2} \rho_{b2} \mathbf{v}_{b2} \cdot d\mathbf{A}_2 + \int_{A_3} \rho_{b3} \mathbf{v}_{b3} \cdot d\mathbf{A}_3 + \int_V \frac{\partial \rho_b}{\partial t} dV = \frac{\partial m_{gen}}{\partial t}, \quad (5)$$

where the first three terms that are broken down from the first term on the left side of (4) represent the bulk mass flux into or out of the specified zone by crossing three areas covering the specified zone. The three areas are $A_1 = 2\pi rh$, the circumferential vertical surface of the cylindrical aquifer; $A_2 = \pi r^2$, the area of the cylindrical bottom interfacing the underlying confining bed at $z=0$; and A_3 , the surface of the upper boundary at $z=h(r, t)$ (Fig. 2). The term $\mathbf{v}_{bi}$ ($i=1, 2$ and 3) is the bulk mass flux crossing the area A_i ($i=1, 2$, and 3), h is the hydraulic head in the unconfined aquifer, and z is the vertical coordinate.

If the following assumptions are invoked, (1) water injected into or pumped from the aquifer is considered as a source or a drain of water mass within V [i.e., $\partial m_{gen}/\partial t \neq 0$], (2) the solid grains flow into the specified zone and stop at the well screen [i.e., $v_s(r_w, t)=0$]; (3) the filtration to the water table through land surface above is negligibly small, (4) water and solid individual grains are incompressible, (5) the Dupuit assumption is valid, (6) water flow through the confining bed is vertical, and (7) the net accumulation of mass with time within the specified zone is caused by an increase of the mass of solids that more than offsets the mass of water released at

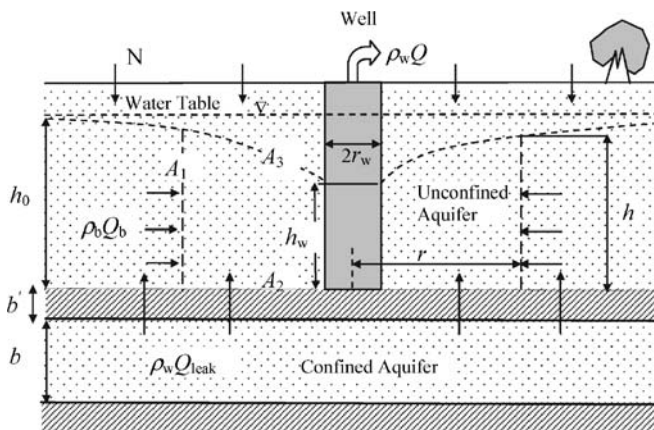


Fig. 2 Radial bulk flux in an unconfined leaky aquifer with a fully penetrating well

r_w from aquifer storage within the same zone. The net rate of accumulation of mass equals $2\pi hr \rho_s' (1-n)v_s$, where $\rho_s' = \rho_s - \rho_w$, thus, (5) for bulk mass balance within the specified zone (Fig. 2) becomes:

$$-2\pi rh \rho_b v_b(r, t) - \rho_w Q_{leak}(r, t) + 2\pi hr \rho_s' (1-n)v_s(r, t) = -\rho_w Q(t) \quad (6)$$

On the left-hand side of (6), the first term is a product that yields the lateral bulk flow by mass crossing surface A_1 at r . The second product $\rho_w Q_{leak}$ represents water flow by mass into the unconfined aquifer through A_2 , where the leakage $Q_{leak} = (\pi K'/b') \int_0^r \Delta h'/dr^2$, where b' and K' are thickness and vertical conductivity of the confining bed (aquitar), and $\Delta h'(r, t)$ is hydraulic drawdown crossing the confining bed. The third product $2\pi hr \rho_s' (1-n)v_s$ with a positive sign stands for a net increase rate of cumulative mass due to assumptions (2) and (7) listed earlier in this section. The bulk mass flux through A_3 is zero, due to assumption (3a, 3b). On the right side of Eq. (6), the product with a negative sign represents a drain with the rate of water mass pumped from the unconfined aquifer, i.e., $\partial m_{gen}/\partial t = -\rho_w Q(t)$, where $Q(t)$ is the well discharging rate by volume (a positive sign for recharge). The signs in front of the first two terms on the left side of (6) are determined by the dot product of $\mathbf{v}_b \cdot d\mathbf{A} = \cos(\alpha) v_b dA$, where α is the angle between vectors $\mathbf{v}_b$ and $d\mathbf{A}$ in (5). As all normal vectors on the three surfaces point outward, the terms related to influent crossing A_1 or A_2 have negative signs because $\cos(180^\circ) = -1$. Consequently, the bulk velocity can be found from (6) in the following form:

$$v_b(r, t) = \frac{\rho_w}{\rho_b} \left[\frac{Q(t) - Q_{leak}(t)}{2\pi hr} + (1-n)(G-1)v_s \right] \quad (7)$$

$(r_w \leq r < \infty)$

where $G (\equiv \rho_s/\rho_w)$ is the specific gravity of individual soil grains. From conservation of water mass flow alone or bulk mass flux in (6), the following relation can be derived:

$$nv_w + (1-n)v_s = \frac{Q(t) - Q_{leak}(t)}{2\pi hr} \quad (8)$$

$(r_w \leq r < \infty)$

where v_w is the velocity of water in the radial direction. The left side of (8) actually represents the radial bulk volume flux $q_b [\equiv nv_w + (1-n)v_s = v_s + q]$ defined by Helm (1994b), where $q [\equiv n(v_w - v_s)]$ is the radial specific discharge.

Radial-specific discharge based on mass balance

Equating the radial component of $\mathbf{v}_b$ in (3a, 3b) and the same term in (7) and considering (8) give the radial specific discharge:

$$q(r, t) = \frac{Q(t) - Q(t)_{leak}}{2\pi rh} - v_s(r, t) \quad (9)$$

which is based on mass balance and written in terms of aquifer velocity, well pumpage, hydraulic head, and aquifer leakance. An alternative way to reach (9) is to apply the relation $nv_w + (1 - n)v_s = v_s + q$, as mentioned earlier, to the left side of (8).

Cooper (1966) points out (as Gersevanov (1937) did earlier) that the governing equations of groundwater flow, such as the Darcy law, express that the flow of water is relative to the skeletal frame. He further demonstrates that the skeletal frame deforms during transient flow. The standard nonequilibrium groundwater flow equations for drawdown are valid only in terms of a deforming coordinate frame that remains fixed in the deforming aquifer. Helm (1994b) took Cooper's analysis a quantitative step further. Based on Eq. (3b), Helm showed that for boundary conditions that force v_s eventually approaches zero throughout an aquifer zone of interest, the specific discharge q must correspondingly approach volume bulk flux q_b . Hence, the value of q_b can be approximated throughout the zone of interest simply by solving for q from the standard transient groundwater flow equation and then setting q_b equal to the limiting value of q as time increases without limit. Transient v_s itself is estimated by subtracting the transient value of q from this calculated value for q_b in accordance with Eq. (3b). Though approximate, this v_s value is more realistic than the standard assumption of a rigid aquifer. In the deforming aquifer, the term $\rho_b v_b$ can reduce to $\rho_w q$ that is a rigorous result and is consistent with Cooper's and Helm's analysis.

Darcy-Gersevanov law for radial flow

For saturated and porous sedimentary materials, if inertial forces, namely, accelerations of water, solids, and their difference (i.e., $\mathbf{a}_w$, $\mathbf{a}_s$ and $\mathbf{a}_w - \mathbf{a}_s$) are negligibly small when compared to viscous drag and seepage forces, the relation between viscous drag and driving forces reduces to the Darcy-Gersevanov Law (Li and Helm 1998):

$$\mathbf{q} = -\mathbf{K} \nabla h \quad (10)$$

where $\mathbf{K}$ is a tensor of hydraulic conductivity and ∇ is a differential operator of gradient. If the aquifer velocity $\mathbf{v}_s$ is negligibly small when compared to the water velocity

$\mathbf{v}_w$, Eq. (10) further reduces to Darcy's law. In an axisymmetric coordinate system, $\mathbf{q}$ can be expressed as $\mathbf{q} = \{q_r, q_\theta, q_z\}$. In particular, in the radial direction the Darcy-Gersevanov's law can be written as:

$$q = n(v_w - v_s) = -K \frac{\partial h}{\partial r}, \quad (11)$$

where K (or K_{rr}) and q (or q_r) are the radial hydraulic conductivity and the radial specific discharge as defined earlier. Similarly, the subscript r is dropped for simplicity of notation. Although the acceleration terms $\mathbf{a}_w$, $\mathbf{a}_s$ and $\mathbf{a}_w - \mathbf{a}_s$ in the Darcy-Gersevanov law are assumed to be negligibly small, they cannot be zero individually so that the unsteady flow and transient aquifer movement can be described (Li and Helm 1998).

Analytic unsteady drawdown in an unconfined leaky aquifer

For a single, fully penetrating well installed within a unconfined aquifer (Fig. 2), the governing equation for radial unsteady flow in terms of the hydraulic head with leakance from the confining bed below and filtration through the unsaturated zone above can be defined as (Bear 1979):

$$\begin{aligned} \frac{\partial}{\partial r} \left(Kh \frac{\partial h}{\partial r} \right) + \frac{1}{r} \left(Kh \frac{\partial h}{\partial r} \right) + N + \frac{h_c - h}{\sigma} \\ = S \frac{\partial h}{\partial t}, \end{aligned} \quad (12)$$

where h is the hydraulic head in the pumped unconfined aquifer, h_c is the hydraulic head in the unpumped confined aquifer, S is the specific yield, N is the infiltration to the water table, the term $(h_c - h)/\sigma$ represents the vertical leakance through the semi-permeable confining layer, where σ ($\equiv b'/K'$) is the coefficient of leakance, and b' and K' are the thickness and vertical hydraulic conductivity of the confining bed (aquitar), respectively. If the following assumptions are applied to the flow equation: (1) storage within the confining layer is negligible, (2) the aquifer is homogeneous, (3) no filtration flows into the water table, and (4) the hydraulic head h_c in the confined aquifer either equals the constant initial head h_0 or changes with the drawdown square s^2 within the unconfined aquifer, the unsteady flow equation (12) can be linearized for a complete mathematical description as follows (Bear 1979):

$$\frac{\partial^2 s'}{\partial r^2} + \frac{1}{r} \frac{\partial s'}{\partial r} - \frac{s'}{\lambda^2} = \frac{S'}{T_0} \frac{\partial s'}{\partial t}, \quad (13a)$$

and the initial condition:

$$s'(r, 0) = 0, \quad (13b)$$

and boundary conditions:

$$s'(\infty, t) = 0, \quad (13c)$$

$$Q(t) = -2\pi T_0 \lim_{r=r_w \rightarrow 0} r \partial s' / \partial r, \quad (13d)$$

where λ is the leaky factor and defined by $\lambda \equiv (b'T_0/K')^{1/2}$; s' equals $s - s^2/2h_0$, where $s = h_0 - h$ is the drawdown; h_0 and h are hydraulic heads in the unconfined aquifer at times $t=0$ and $t=t$, where h equals h_0 , when r approaches ∞ ; T_0 ($\equiv Kh_0$) is the initial transmissivity of the unconfined aquifer; S' is an "apparent" storage coefficient and is defined as $S' \equiv Sh_0/(h_0 - s)$, where S is the specific yield of the unconfined aquifer. For an accurate solution, (13a) needs S' to be an approximate constant that requires either an approximate constant S when $s \ll h_0$ or an approximate constant ratio $S/h \approx k_0$, where k_0 is a constant though both S and h can individually be changeable (Bear 1979). Assumption (4) in this section indicates two ways to allow the term $(h_c - h)/\lambda^2$ in (12) to become s'/λ^2 in (13a). One way is simply to assume $h_c - h \approx s'$ that requires h_c to change with drawdown s^2 within the pumped unconfined aquifer. The other way is to assume a constant h_c to lead $h_c - h \approx s'$, where first it is to assume $h_c = h_0$ (i.e., $h_c - h = s$), and then it is to assume $s \ll 2h_0$ (i.e., $s^2/2h_0 \ll s$) so that $s' = s - s^2/2h_0 \approx s$. In the similar way approached by Carslaw and Jaeger (1959), solving (13a) for s' with the initial and boundary conditions from (13b) to (13d) results in the analytic solution as shown below:

$$s' = \frac{1}{4\pi T_0} \int_0^t \frac{Q(t-\tau)}{t-\tau} e^{-\left[\frac{r^2 S'}{4T_0(t-\tau)} + \frac{(t-\tau)T_0}{S'\lambda^2}\right]} d\tau, \quad (14)$$

where τ is a dummy variable of integration. If the right side of (14) is considered as a well function, the relation between s' and this well function is linear. The relation between the hydraulic head h and the well function, however, is not, because the term $s' (\equiv s - s^2/2h_0)$ in (14) can be rewritten as:

$$h_0^2 - h^2 = \frac{1}{2\pi K} \int_0^t \frac{Q(t-\tau)}{t-\tau} e^{-\left[\frac{r^2 S'}{4T_0(t-\tau)} + \frac{(t-\tau)T_0}{S'\lambda^2}\right]} d\tau, \quad (15a)$$

where the hydraulic head versus the well function is nonlinear, due to the term h^2 . Therefore, Eq. (15a) sets

itself distinctively apart from that in a confined aquifer. For convenience of applications of the analytic drawdown in future investigations, (15a) can be alternatively written as:

$$h = \left\{ h_0^2 - \frac{1}{2\pi K} \int_0^t \frac{Q(t-\tau)}{t-\tau} e^{-\left[\frac{r^2 S'}{4T_0(t-\tau)} + \frac{(t-\tau)T_0}{S'\lambda^2}\right]} d\tau \right\}^{1/2}. \quad (15b)$$

The nonlinear relation between the hydraulic head and the well function in (15a) or (15b) can be further linearized if the drawdown is significantly small. Particularly when $s < 0.02 h_0$, for example, the analytic drawdown in (15b) simplifies to a linear expression as shown below (Jacob 1950):

$$h - h_0 = \frac{1}{4\pi T_{avg}} \int_0^t \frac{Q(t-\tau)}{t-\tau} e^{-\left[\frac{r^2 S'}{4T_0(t-\tau)} + \frac{(t-\tau)T_0}{S'\lambda^2}\right]} d\tau, \quad (15c)$$

where $T_{avg} = Kh_{avg}$ denotes the average transmissivity, and $h_{avg} = (h_0 + h)/2$ is the average hydraulic head and is approximately constant. To reach (15c), $s' = (h_{avg}/h_0)s$ can be applied either to (13a) to linearize the governing equation or to (15a) to linearize the hydraulic head h . The analytic drawdown solution in (15c) appears similar to that in a confined leaky aquifer if T_{avg} is replaced by the transmissivity of the confined aquifer (Li 2006, 2007). This result implies that when $s < 0.02 h_0$, an unconfined leaky or nonleaky aquifer in response to drawdown in (15c) will behave similarly as does a confined aquifer. Therefore, results and analysis for the confined aquifer can be simply applied to the unconfined case when $s \ll 0.02 h_0$.

Analytic solutions of aquifer radial transient movement

To understand how an unconfined aquifer movement is affected by hydraulic forces and is governed by the Darcy-Gersevanov law, the analytic solutions in the velocity and displacement fields need to be found.

Radial velocity of an unconfined leaky aquifer

Taking the derivative of the hydraulic head in either (15a) or (15b) with respect to the radial distance r yields:

$$-K \frac{\partial h}{\partial r} = \frac{Q(\xi)}{2\pi r h} e^{-\left(\xi + \frac{r^2}{4\lambda^2 \xi}\right)}, \quad (16)$$

where $\xi (\equiv r^2 S'/4T_0 t)$ is a dimensionless variable. Actually, (16) represents the relation of the Darcy-

Gersevanov law in terms of the well flow rate $Q(t)$ and the transient hydraulic head $h(r, t)$ that is presented in (15b). It is interesting to note that from (14), (16) can be alternatively expressed by:

$$\frac{\partial s''}{\partial r} = \frac{Q(\xi)}{2\pi r T_0} e^{-\left(\xi + \frac{r^2}{4\lambda^2 \xi}\right)} \quad (17)$$

Equating the radial component of the specific discharge q that appears in (9) and (11) and considering (16) result in the aquifer velocity as:

$$v_s(\xi, t) = \frac{1}{D_0 h} \left\{ Q \left[1 - e^{-\left(\xi + \frac{r^2}{4\lambda^2 \xi}\right)} \right] - Q_{leak} \right\}, \quad (18)$$

where $D_0 = 2\pi r$.

Equation (18) represents the transient velocity of the unconfined aquifer in response to variable well-flow rates. The analytic radial velocity in (18) indicates: (1) that aquifer radial movement slows down when the hydraulic head h increases, (2) that aquifer horizontal movement speeds up when pumpage $Q(t)$ increases, (3) that aquifer horizontal movement decreases if leakance increases. The next task is to find aquifer cumulative displacement from the aquifer velocity in (18).

Radial cumulative displacement of an unconfined leaky aquifer

As mentioned in the previous section, the well recharge-discharge rate $Q(t)$ is a function of time. Because aquifer velocity is defined as a derivative of displacement with respect to time, integrating aquifer velocity (18) over time yields an aquifer cumulative displacement in the radial direction as:

$$\begin{aligned} u_s(r, t) &= \int_0^t v_s(r, t) dt \\ &= \frac{D_1}{D_0} \int_{\xi}^{\infty} \frac{Q \left[1 - e^{-\left(m + \frac{B^2}{m}\right)} \right] - Q_{leak}}{m^2 h} dm \end{aligned} \quad (19)$$

where $D_1 = r^2 S' / 4T_0$ and the dimensionless leaky factor $B = r/4\lambda$, m is a dummy variable of integration and an assumption of no initial displacement is applied, i.e., $u_s(r, 0) = 0$. The solution (19) represents the radial transient displacement of an unconfined and leaky aquifer in response to the well-flow rate $Q(t)$. It should be pointed out that the leakance $Q_{leak}(r, t)$ is also a function of the drawdown s along the radial direction. If

$\Delta h' = h_c - h \approx s'$ and $r_w \approx 0$ as assumed earlier are invoked, recalling Q_{leak} defined after (6) and s' in (14) results in the leaky flow $Q_{leak} = \int_{\xi}^{\infty} Q e^{-m} (1 - e^{-B^2/m}) dm$.

Furthermore, using definitions of aquifer strain and strain rate, i.e., $\varepsilon \equiv \partial u / \partial r$ and $\partial \varepsilon / \partial t \equiv \partial v / \partial r$, the solutions in the strain field can be derived from (18) and (19) by taking derivatives of velocity and displacement with respect to the radial distance r .

Discussion of analytic solutions of an unconfined leaky aquifer

In this section, the analytic solutions of Eqs. (18) and (19) for the unconfined leaky aquifer are discussed along with various special cases so that aquifer radial movement with simplified pumping conditions can be described. The following four special cases are discussed to understand the generality of the solutions:

1. If the well pumpage is constant [i.e., $Q(t) = Q_0$], the well function, i.e., the term on the right side of (14), becomes a form parallel to the Hantush-Jacob well function (Hantush and Jacob 1955) of a confined leaky aquifer for large times. Consequently, (18) and (19) reduce to:

$$v_s(\xi) = \frac{Q_0 \left\{ \left[1 - e^{-\left(\xi + \frac{B^2}{\xi}\right)} \right] - \int_{\xi}^{\infty} e^{-m} (1 - e^{-B^2/m}) dm \right\}}{D_0 h}, \quad (20a)$$

$$u_s(\xi) = \frac{D_1 Q_0}{D_0} \int_{\xi}^{\infty} \frac{\left[1 - e^{-\left(m + \frac{B^2}{m}\right)} \right] - \int_m^{\infty} e^{-u} (1 - e^{-B^2/u}) du}{m^2 h} dm, \quad (20b)$$

where m and u are dummy variables of integrations. The expression of Q_{leak} evaluated after (19) and the ratio $Q_{leak}/Q_0 = \int_m^{\infty} e^{-m} (1 - e^{-B^2/m}) dm$ are applied to (20a) and (20b). This special case is emphasized and discussed in the following sections.

2. If leakance is significantly large and has a value close to the well pumpage [$Q \approx Q_{leak}$], (18) and (19), respectively, become:

$$v_s(\xi) = \frac{Q(\xi)}{D_0 h} e^{-\left(\xi + \frac{B^2}{\xi}\right)} \quad (21a)$$

$$u_s(\xi) = \frac{D_1}{D_0} \int_{\xi}^{\infty} \frac{-Q(m)}{m^2 h} e^{-\left(m + \frac{B^2}{m}\right)} dm \quad (21b)$$

3. If there is no leaky flow through the semi-permeable confining bed, all terms related to leakance are dropped. The well function in (14) reduces a form similar to the Muskat well function (Muskat 1937; Carslaw and Jaeger 1959) for a confined nonleaky aquifer with variable pumpage. Thus, (21a) and (21b) further reduce to:

$$v_s(\xi) = \frac{Q(t)}{D_0 h} (1 - e^{-\xi}) \quad (22a)$$

$$u_s(\xi) = \frac{D_1}{D_0} \int_{\xi}^{\infty} \frac{Q(m)(1 - e^{-m})}{m^2 h} dm \quad (22b)$$

Equations (22a) and (22b) are the solutions in the velocity and displacement fields of an unconfined and nonleaky aquifer.

4. If the well-flow rate is constant [i.e., $Q(t)=Q_0$], for no leakance the well function on the right side of (14) reduces to the one similar to the Theis well function (Theis 1935) for a confined nonleaky aquifer with constant pumpage. Thus, (22a) and (22b) accordingly simplify to:

$$v_s(\xi) = \frac{Q_0}{D_0 h} (1 - e^{-\xi}) \quad (23a)$$

$$u_s(\xi) = \frac{D_1 Q_0}{D_0} \int_{\xi}^{\infty} \frac{(1 - e^{-m})}{m^2 h} dm \quad (23b)$$

It is important to note that though the general solutions (18) and (19) and their special cases from (20a) to (23b) appear similar to those of a confined leaky or nonleaky aquifer because the effect of the hydraulic head h in an unconfined aquifer is similar to the effect of aquifer thickness b of a confined aquifer, the physical distinction and difference between the two sets of solutions are substantial. This is because h in the unconfined aquifer is a variable and governed by the nonlinear relation (15a) or (15b), whereas b simply is a constant. Furthermore, as discussed earlier in (15c), for the special case of $s \ll 0.02 h_0$ the term T_{avg} can be replaced by the transmissivity T of the confined aquifer. Thus, the solutions (18) and (19) and their special cases appear identical to those of the confined aquifer. This result suggests that the previous solutions for a confined leaky aquifer developed by Li (2006, 2007) can be applied to the unconfined leaky aquifer when drawdown is smaller than 2% of the initial head h_0 . Again, for simplicity of notation, the subscript s in the expressions from (18) and (23b) is dropped in the rest of this paper.

Moreover, in this paper, the phrases “aquifer movement”, “aquifer velocity”, and “aquifer displacement” can be interchangeably used by “movement of aquifer matrix”, “velocity of aquifer matrix”, and “displacement of aquifer

matrix”. In the next section, the effect of the vertical leaky flow on aquifer movement is discussed.

Discussion and analysis of the solutions

Leaky impact on aquifer velocity

In this section, the effect of leakance on the aquifer radial movement including velocity and displacement are investigated. Aquifer transient velocity (20a) can be alternatively expressed in a dimensionless form:

$$v^*(t^*) = \frac{\left\{ \left[1 - e^{-\left(\frac{1}{4t^*} + 4B^2 t^*\right)} \right] - \int_{1/4t^*}^{\infty} e^{-m} \left(1 - e^{-B^2/m} \right) dm \right\}}{h^*(t^*)} \quad (24a)$$

where v^* ($\equiv D_0 h_0 v / Q$) is the dimensionless velocity; t^* ($\equiv t T_0 / r^2 S$) is the dimensionless time; the dimensionless hydraulic head h^* ($\equiv h / h_0$) in terms of t^* can be derived from (15b) and given in (27a) discussed in a later section.

For $Q=128 \text{ m}^3/\text{day}$, $K=1.73 \text{ m/day}$, and $h_0=50 \text{ m}$, the dimensionless relation of velocity versus time in (24a) is shown in Fig. 3a and b. Figure 3a indicates that the transient aquifer velocity in the radial direction is responsive to leakance. For $B < 0.01$, for instance, the radial aquifer speed decreases with pumping time. In contrast, for $B \geq 0.01$, aquifer movement slows down initially and then speeds up with time later. Particularly, when $t^* > 10$, aquifer velocity quickly increases with the leaky parameter B varying from 0 to 0.03. This increase means that aquifer movement is significantly affected by leakance in a time domain. Figure 3b shows that the aquifer velocity versus the pumping time is significantly affected by the initial water head if $h_0 < 6 \text{ m}$ and $B=0.01$. Figure 3b illustrates that the aquifer moving speed increases rapidly particularly when time $t^* > 10$ and $h_0 < 6 \text{ m}$. To investigate the effects of leakance on aquifer movement along the radial direction, (24a) can be rewritten as the following dimensionless expression:

$$v^*(r^*) = \frac{\left[1 - e^{-\left(\frac{r^{*2}}{4} + \frac{4B^2}{r^{*2}}\right)} \right] - \int_{r^{*2}/4}^{\infty} e^{-m} \left(1 - e^{-B^2/m} \right) dm}{r^* h^*(r^*)} \quad (24b)$$

where $r^* = r(S'/T_0 t)^{0.5}$, $v^* = 2\pi h_0 v / [Q(S'/T_0 t)^{0.5}]$, and $h^*(r^*)$ is given in (27b) that is identical to (27a) except that the lower limit $1/4t^*$ of integration is replaced by $r^{*2}/4$. The dimensionless relation of velocity versus radial distance is drawn in Fig. 3c using the same parameters applied in (24a).

Figure 3c demonstrates that the radial aquifer velocity increases rapidly after the pump is turned on, and then slows down after the aquifer moving speed reaches a maximum $v^*_{\max}$ equaling 0.32 at the location $r^*_{\max}=2.25$ that horizontally separates the aquifer into two zones that will be discussed in detail later.

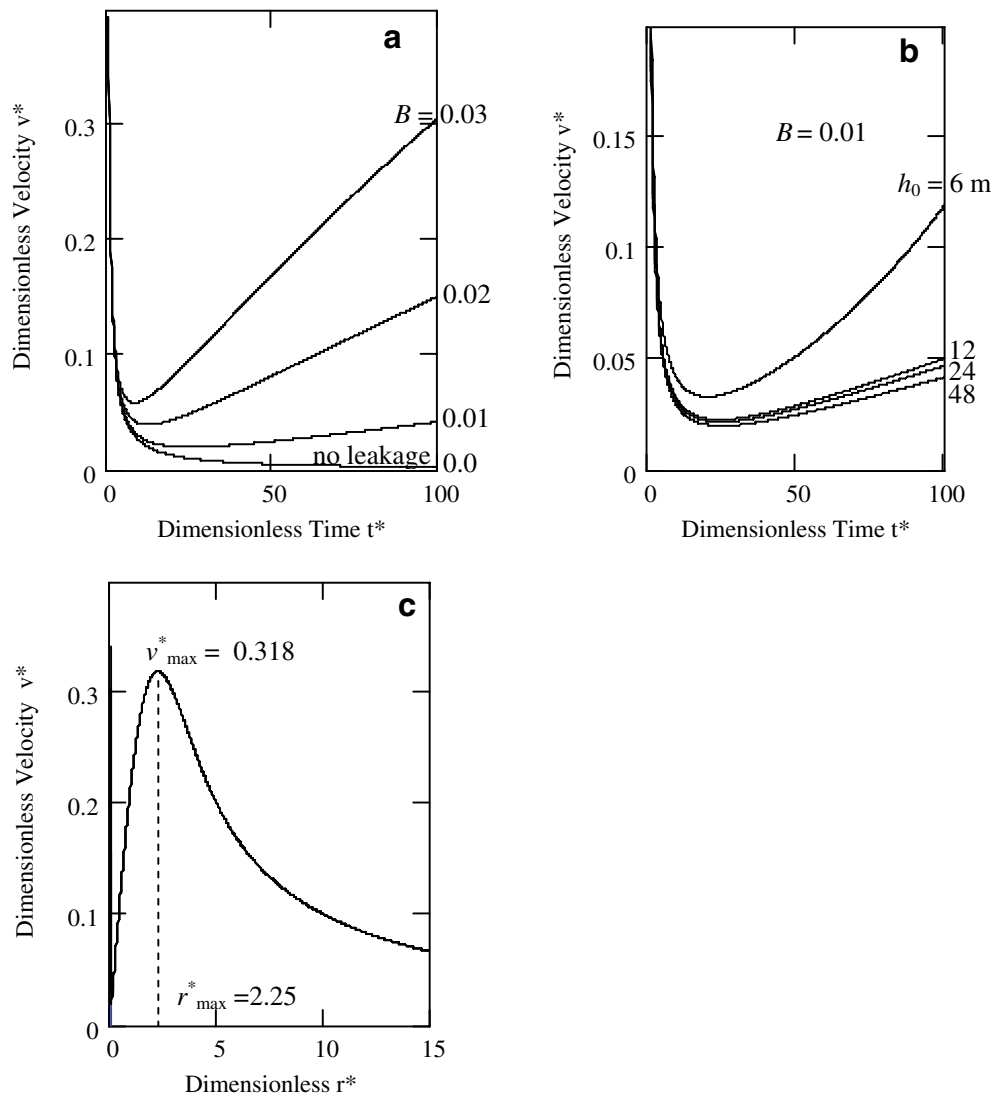


Fig. 3 a Dimensionless velocity v^* versus time t^* with B ; b Dimensionless velocity v^* versus time t^* with h_0 ; c Dimensionless velocity v^* versus radial distance r^*

Leaky effect on aquifer displacement

Similarly to the discussion above, rewriting aquifer displacement (20b) with dimensionless variables yields:

$$u^*(t^*) = \int_{1/4t^*}^{\infty} \frac{\left[1 - e^{-\left(m + \frac{B^2}{m}\right)}\right] - \int_m^{\infty} e^{-u} \left(1 - e^{-\frac{B^2}{u}}\right) du}{m^2 h^*(m)} dm \quad (25a)$$

where the dimensionless displacement $u^* = (D_0/D_1)uh_0/Q$, where Q is assumed to be constant; $h^*(m)$ has the same form as (27a) or (27b) except that the lower limit of integration is replaced by m . Equation (25a) is plotted in Fig. 4a and b using the same values of Q , K , and h_0 applied in Fig. 3a,b, and c. Figure 4a illustrates that after the pumping starts, the radial aquifer displacement increases dramatically during the period $0 < t^* < 10$ when leakance has little effect on the aquifer deformation. When $t^* > 10$,

however, aquifer displacement is significantly affected by the leaky parameter B . In particular, for $B > 0.01$, each curve of aquifer displacement versus time deviates quickly from the curve without leakance. Figure 4b exhibits a similar result that is indicated in Fig. 3b, namely, that aquifer deformation increases significantly with time if the initial water head h_0 in the unconfined aquifer is less than 8 m.

To investigate the leaky effects on aquifer radial deformation needs the following dimensionless expression in terms of the radial distance r :

$$u^*(r^*) = \int_{r^{*2}/4}^{\infty} \frac{\left[1 - e^{-\left(m + \frac{B^2}{m}\right)}\right] - \int_m^{\infty} e^{-u} \left(1 - e^{-\frac{B^2}{u}}\right) du}{m^2 h^*(m)} dm \quad (25b)$$

where $u^* = 4\pi u(h_0/Q)/(S't/T_0)^{0.5}$. The relation of the dimensionless displacement versus the dimensionless

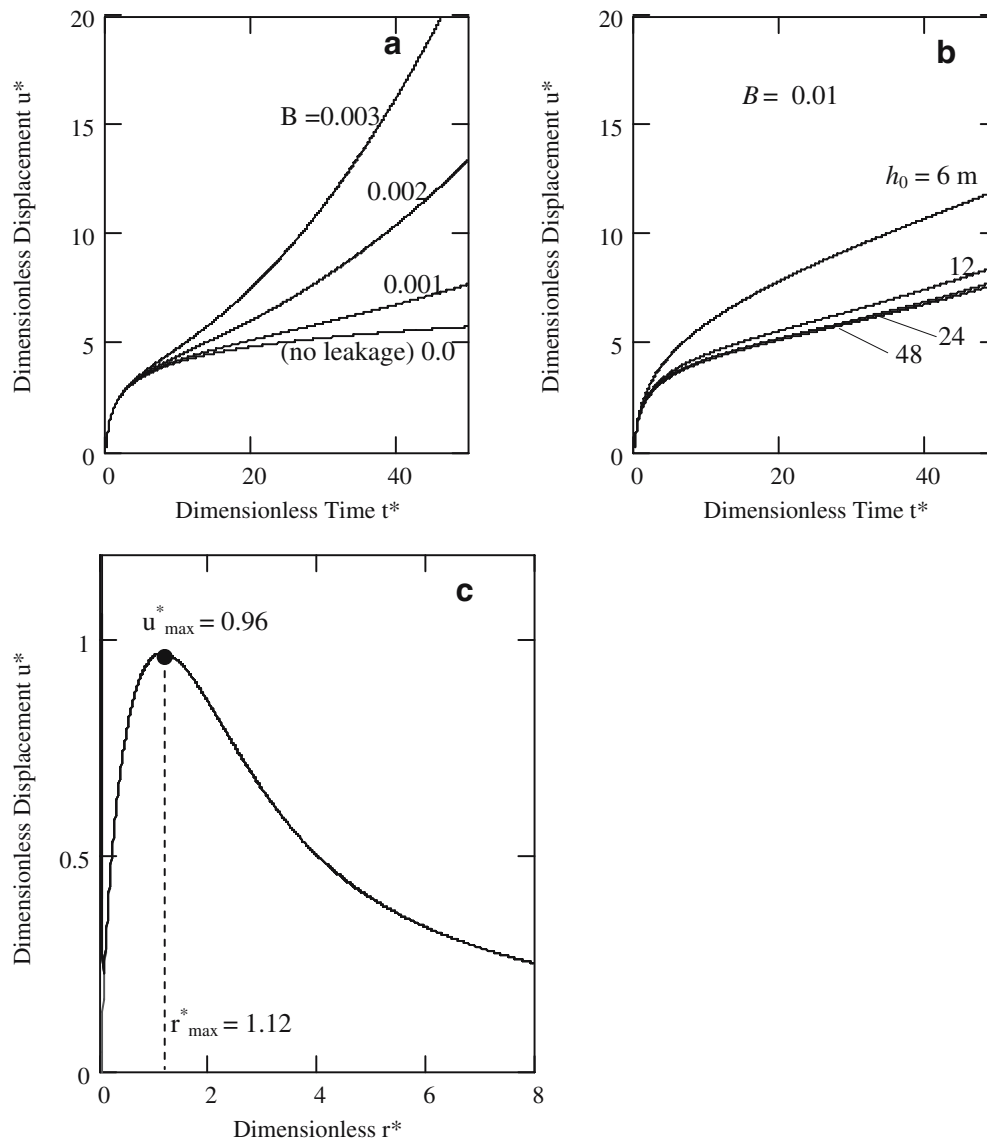


Fig. 4 a Dimensionless displacement u^* versus time t^* with B ; b Dimensionless velocity u^* versus time t^* with h_0 ; c Dimensionless displacement u^* versus radial distance r^*

radial distance is shown in Fig. 4c, which illustrates that radial aquifer displacement increases dramatically after the pump is turned on, and then decreases after a maximum radial deformation $u^*_{\max}$ ($= 0.96$) occurs at the radial location $r^*_{\max}$ ($= 1.12$) that also divides the aquifer into two zones: the deformation compression zone and the deformation extension zone (Helm 1994b). From Fig. 4c, the dimensionless relation of the aquifer displacement versus the radial distance does not have effects on leakance (or the leaky parameter B). This result is also similar to the dimensionless relation of the aquifer velocity versus the radial distance presented in Fig. 3c. As Li pointed out in his studies (2006, 2007), both velocity and displacement of the unconfined leaky aquifer are very responsive to leakance

(i.e., B) in the time domain but not in space because aquifer movement along the radial direction is irrelevant to leakance as the leaky term B^2/m in (18) and (19), inherently, is not a function of r , but a function of time only, i.e., $B^2/m = tT_0/S'\lambda^2$.

Impact factor of leakance on aquifer radial movement

To describe leaky effects quantitatively on aquifer movement, impact factors can be introduced. The impact factors of leakance are defined using the ratios of velocity and displacement with leakance to those without leakance.

Namely, the ratios of (20a) to (23a) and (20b) to (23b) are given as below:

$$I_{vel} = \left[\left(1 - e^{-\left(\xi + \frac{B^2}{\xi} \right)} \right) - \int_0^\infty e^{-m} \left(1 - e^{-B^2/m} \right) dm \right] / (1 - e^{-\xi}), \quad (26a)$$

$$I_{disp} = \left[\frac{\int_\xi^\infty \left(1 - e^{-\left(m + \frac{B^2}{m} \right)} \right) - \int_0^\infty e^{-u} \left(1 - e^{-B^2/u} \right) du}{m^2 h^*(m)} dm \right] / \left[\frac{\int_\xi^\infty (1 - e^{-m})}{m^2 h^*(m)} dm \right], \quad (26b)$$

where the notation I denotes the leaky effect on the radial movement of the unconfined aquifer; subscripts vel and disp represent the leaky impact factors of aquifer velocity

and displacement individually. It should be pointed out that the solutions (20a) and (20b) require no vertical component of compression. If Helm's strategy for approximating q_b as discussed earlier is applied in order to approximate a vertical component of v_s , an equilibrium $Q_{leak}(t, r) = Q_{leak}(\infty, r)$ and $R = \int_0^\infty e^{-u} (1 - e^{-B^2/u}) du$ can be applied to (26a) and (26b) above.

The impact factors I_{vel} and I_{disp} versus dimensionless time for small leakance (e.g., $B = 0.001 = 0.005$) are drawn in Fig. 5a and b using the parameters $h_0 = 50$ m, $Q = 2.38$ m³/day, and $K = 1.73$ m/day. Figure 5a and b indicate that: (1) dimensionless relations of I_{vel} and I_{disp} versus time t^* show a similar pattern as is shown above. Namely, the impact factor I considerably increases with time t^* and leakance; (2) the unconfined aquifer with leakance moves faster than that without leakance does as I_{vel} is equal to or greater than 1; (3) the unconfined aquifer with leakance is more deformable than that without leakance is as I_{disp} is also equal to or larger than 1; (4) aquifer velocity is more responsive to time when compared to aquifer displace-

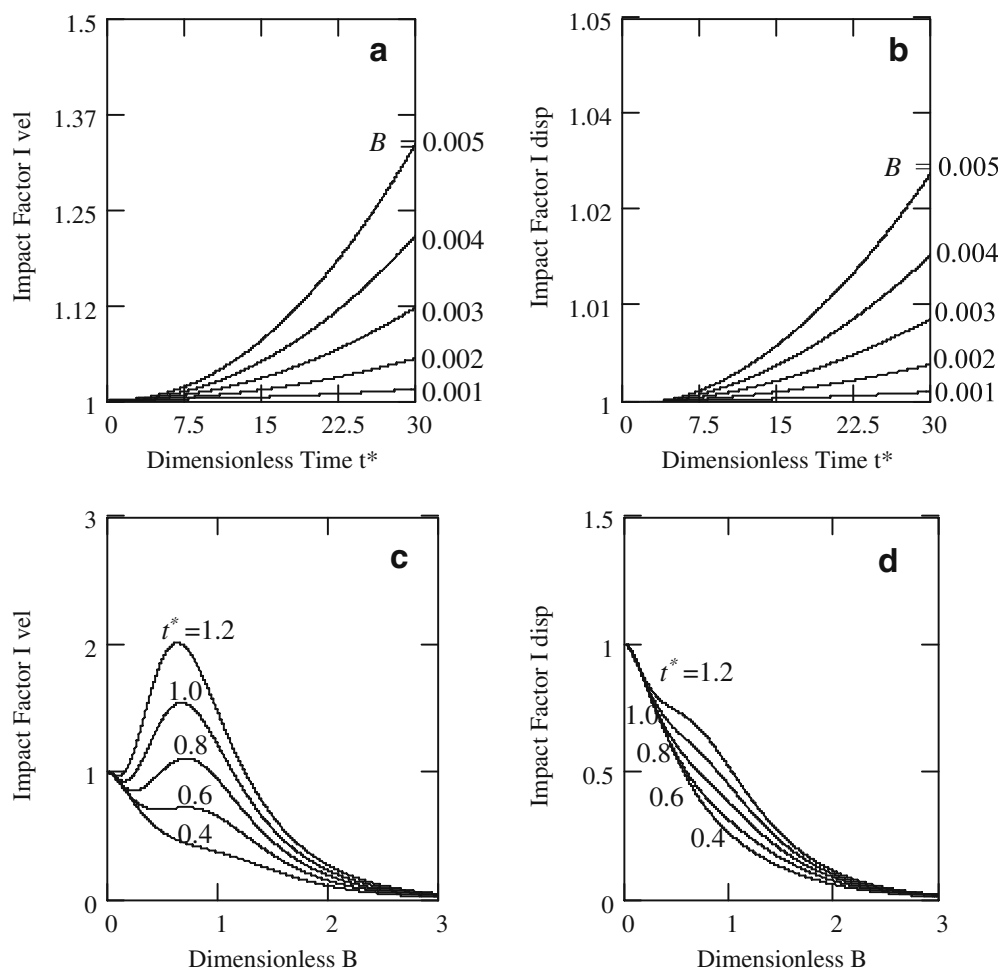


Fig. 5 a Impact factor I_{vel} versus time t^* with B ; b Impact factor I_{dis} versus time t^* with B ; c Impact factor I_{vel} versus B with time t^* ; d Impact factor I_{dis} versus B with time t^*

ment as I_{vel} is larger than I_{disp} for the same value of the parameter B at a given time. For instance, $I_{vel}=1.33$ and $I_{disp}=1.03$ at $t^*=30$ and $B=0.005$ indicate 33% changes in aquifer velocity versus 3% changes in aquifer displacement due to leakage flowing into the pumped unconfined aquifer. This result suggests that for the given leakage, the vertical leaky flow has more of an effect on the moving aquifer speed than on aquifer deformation.

For large leakage, the impact factors I_{vel} and I_{disp} versus the dimensionless leaky factor B are shown in Fig. 5c and d using the same parameters. Figure 5c shows that for $B<1$, when $t^*>0.8$, the unconfined aquifer moves faster with leakage than without leakage as $I_{vel}<1.0$. When $t^*<0.8$, however, the unconfined aquifer moves slower with leakage than without leakage as $I_{vel}>1.0$, and this result is also true for $B>1.5$ during any pumping period. Figure 5d indicates that the aquifer deformation decreases with increasing leakage (or B) and that the unconfined aquifer always deforms less with leakage than without leakage because $I_{disp}<1.0$.

Movement of an unconfined leaky aquifer can be quantitatively evaluated by velocity and displacement of an unconfined nonleaky aquifer times the impact factors discussed in this section.

Impact of initial hydraulic head on drawdown in an unconfined aquifer

As discussed earlier, the hydraulic head in (15a) or (15b) has a nonlinear relation with the well function in the unconfined leaky aquifer. Here, how the nonlinear relation responds to time, radial distance, and the initial hydraulic head as (15a) results in the solutions (18) and (19) indicating substantial difference from Li's (2007) solutions of a confined aquifer are investigated. Recalling

(15b) and rewriting it alternatively in terms of the dimensionless head h^* ($\equiv h/h_0$) yield:

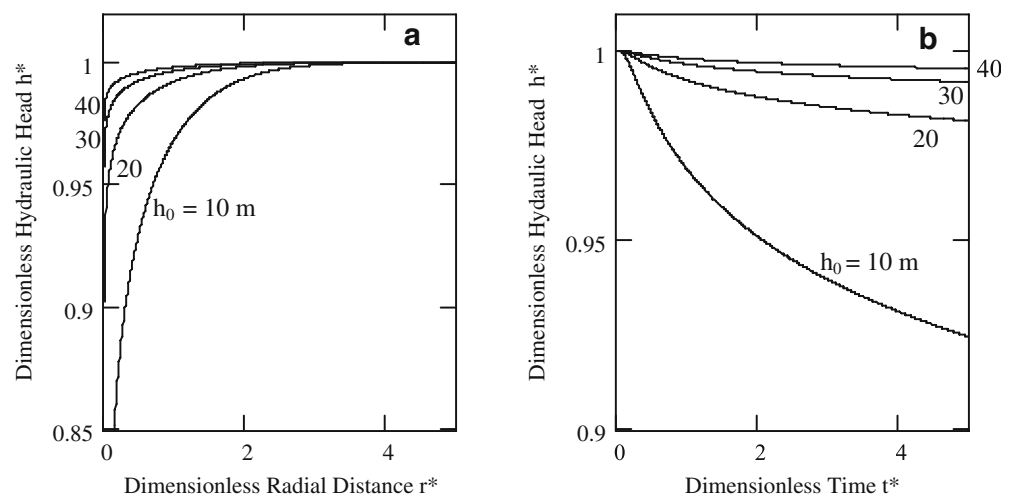
$$h^*(t^*) = \left\{ 1 - \frac{1}{2\pi h_0^2 K} \int_{1/4t^*}^{\infty} \frac{Q}{m} e^{-\left(m + \frac{B^2}{m}\right)} dm \right\}^{1/2} \quad (27a)$$

$$h^*(r^*) = \left\{ 1 - \frac{1}{2\pi h_0^2 K} \int_{r^{*/4}}^{\infty} \frac{Q}{m} e^{-\left(m + \frac{B^2}{m}\right)} dm \right\}^{1/2} \quad (27b)$$

The dimensionless relations of h^* versus t^* and r^* in (27a) and (27b) are drawn in Fig. 6a and b using the same set of parameters (i.e., $Q=128$ m³/day, and $K=1.73$ m/day).

The following conditions are noted from Fig. 6a and b: (1) the relations of the hydraulic head versus pumping time and radial distance are affected by the initial hydraulic head h_0 ; (2) when $h_0>50$ m, the drawdown s ($=h_0-h$) reduces and quickly approaches zero, which suggests that the aquifer movement may not be significantly affected by h^* as h^* quickly approaches 1 within an area near the pumping well (e.g., $r^*<0.5$); (3) when h_0 is larger than 50 m, the drawdown or head deviation from the initial head h_0 does not noticeably change with t^* , which implies that the aquifer movement may not be considerably affected by h^* in the time domain as when $h_0>50$ m, h^* quickly approaches 1 after the pump is turned on.

Fig. 6 a Dimensionless head h^* versus radial distance r^* with B ; b Dimensionless head h^* versus time t^* with B



Tensile stress due to radial movement of an unconfined leaky aquifer

As discussed earlier, Fig. 3c shows that the dimensionless relation between the aquifer velocity and the radial distance has a maximum velocity at the location at $r^* = r_{\max}^*$ that horizontally separates the unconfined aquifer into two areas: a zone where the radial strain increases with time and a zone where the radial strain decreases with time. Where the aquifer radial strain rate (i.e., the slope of the curve in Fig. 3c) is larger than zero (i.e., $\partial v^*/\partial r^* = \partial \varepsilon^*/\partial t > 0$) it is in the strain increasing zone, less than zero ($\partial \varepsilon^*/\partial t < 0$) represents the strain decreasing zone. Radial strain rate equals zero ($\partial \varepsilon^*/\partial t = 0$ or $\varepsilon^* = \text{constant}$) along the boundary between the two zones. If the sediment material that constitutes an aquifer is assumed to behave more realistically, like viscous particles than like an elastic solid, earth fissuring occurs due to the directional change in skeletal strain rate, which correspondingly can be related to viscous tensile stress governed by a poroviscous constitutive law (Li and Helm 1995, 2000). Viscous tensile stress can occur along the boundary between the two zones and within the extension zone of the aquifer. Similarly to the discussion above, the unconfined aquifer in Fig. 4c is also divided into two areas: a zone where the aquifer has deformation compression (i.e., $\partial u^*/\partial r^* = \varepsilon^* > 0$) and a zone where the aquifer has deformation extension (i.e., $\partial u^*/\partial r^* = \varepsilon^* < 0$). The aquifer strain, equal to zero (i.e., $\partial u^*/\partial r^* = \varepsilon^* = 0$), is located along the boundary between the two zones. At the same time, if an elastic stress-strain law is invoked, tensile stress occurs along the boundary between the two zones and within the deformation extension zone when the elastic skeletal strain changes direction.

As soil has little tensile strength, finding locations or zones with potential occurrence of tensile stress is considered as a first approximation of potential analysis of earth fissuring. These locations with tensile stress, as presented by Li (2006, 2007) and Helm (1994b), can be quantitatively determined using the relation $r_{\max}^* = r_{\max}^* / (S'/T_0 t)^{0.5}$ from the definition of r^* given after (24b) in the velocity and displacement fields.

Summary and conclusions

The research results and conclusions from this study can be summarized as follows:

1. New analytic solutions describing transient radial movement of an unconfined leaky aquifer have been developed. The development in the velocity and cumulative displacement fields is based on conservations of linear momentum and mass of saturated bulk flux, the Darcy-Grisevanov law, and the analytic transient drawdown in response to late-time leaky flow and variable well-flow rates in the unconfined leaky aquifer. Several special cases of the solutions are

discussed as the well function in the new solutions can reduce to several simplified forms appearing similar and parallel to those of a confined aquifer, such as the Hantush-Jacob, Muskat (or Carslaw-Jaeger), and Theis well functions, although the physical difference between the solutions of an unconfined and a confined aquifer is substantial.

2. The relation between the hydraulic head and the well function of the unconfined leaky aquifer is nonlinear. This nonlinear relation substantially distinguishes the solutions of an unconfined aquifer from those of a confined aquifer. When either $h_0 > 50$ m or the drawdown $s < 0.02 h_0$ is applied, nonlinear effects on the solution diminishes. Consequently, differences of the solutions between an unconfined and a confined aquifer become insignificant as the relation between the hydraulic head and the well function approximately becomes linear. The previous research results for a confined aquifer, therefore, can be simply applied to an unconfined aquifer.
3. Two leaky impact factors are introduced for analysis of leaky effects on the aquifer transient movement. The results illustrate that leakance is important in the aquifer horizontal movement. For large leakance (e.g., $B=2$), an unconfined aquifer moves slower with leakance than without leakance. Interestingly, when B is less than 1.5, the leaky aquifer moves slower during an early period than a late period. For both short and long pumping period, the unconfined aquifer deforms less with leakance than without leakance once the pump is turned on. The aquifer velocity seems more responsive to leakance when compared to the aquifer cumulative displacement. For small leakance (e.g., $B=0.03$), however, the conclusion can be reverse. For instance, aquifer speed and deformation may increase with an increasing B .
4. Based on the investigations in this paper, some interesting zones discovered in confined aquifers are also found in an unconfined aquifer, such as strain-rate increasing and decreasing zones, and deformation compression and extension zones. Because forces on aquifer skeleton change directions in these zones or along the boundary between the zones, tensile stress may occur. Finding the locations and zones with possible occurrence of tensile stress is considered as the first approximation for risk assessment of earth fissuring caused by well discharge-recharge activity.

In brief, the new solutions presented here are not only useful for analysis of aquifer motion and deformation caused by hydraulic forces that are induced by withdrawal of ground fluids (i.e., water, oil, and gas) but also convenient for risk assessment of earth fissuring.

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