

Statistical analysis of the eigenspace components of the two-dimensional, symmetric rank-two strain rate tensor derived from the space geodetic measurements (ITRF92-ITRF2000 data sets) in central Mediterranean and Western Europe

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SUMMARY

In the deformation analysis with a 2-D (or planar and horizontal), symmetric rank-two deformation tensor in geosciences (geodesy, geophysics and geology), the *eigenspace components* of these random deformation tensors (principal components, principal directions) are of focal interest. With the new development of space-geodetic techniques, such as GPS, VLBI, SLR and DORIS, the components of deformation measures (such as the stress or strain tensor, etc.) can be estimated from their highly accurate regular measurement of positions and change rates and analysed by means of the proper statistical testing procedures. In this paper we begin with a review of the results of statistical inference of eigenspace components of the 2-D symmetric, rank-two random tensor ('random matrix'), that is, the best linear uniformly unbiased estimation (BLUE) of the eigenspace elements and the best invariant quadratic uniformly unbiased estimate (BIQUE) of its variance–covariance matrix. Then the geodynamic setting of the Earth and especially the selected investigated region—the central Mediterranean and Western Europe will be discussed. Thirdly, the ITRF sites are selected according to the history and quality of the ITRF realization series, and the related incremental velocities of selected ITRF sites are computed. Fourthly, the methods of derivation for the 2-D geodetic strain rates are introduced in order to obtain these strain rates from the incremental velocities. In the case study, both BLUE and BIQUE models as well as related hypothesis tests are applied to the eigenspace components of the 2-D strain rate tensor observations in the area of the central Mediterranean and Western Europe, as derived from the ITRF92 to ITRF2000 sequential station positions and velocities. The interpretation and comparison of these results with the geodynamic feature are followed. Furthermore the statistical inference of the eigenspace components provides us with not only the confidence regions of these estimated, but also the visual presentation of the possible magnitude and the directions of the extension and contraction of the strain rate, which is important for the prediction of the tectonic activity including the possible deformation trend and directions.

Key words: crustal deformation analysis, geodynamics, multivariate Gauss–Markov model, random tensor, satellite geodesy, strain rate tensor.

1 INTRODUCTION

For the deformation analysis in the geosciences (geodesy, geophysics and geology), we are often confronted with the problem of a 2-D (or planar and horizontal), symmetric rank-two deformation tensor. Its *eigenspace components* (principal components, principal direction) play an important role in interpreting the geodetic phenomena like earthquakes (seismic deformations), plate motions and plate deformations among others. With the new space geodetic techniques, such as GPS, VLBI, SLR and DORIS, 3-D positions and velocities of network stations have been determined with high accuracy ($\sim$ mm level) from the regular measurement campaigns, which are acknowledged as an accurate and reliable source of information in Earth deformation studies. This fact suggests that the components of a 2-D deformation tensor can be estimated from the high accuracy geodetic data and analysed through the proper statistical testing procedures. According to the *Measurement Axiom*

such an observed 2-D, symmetric (2, 0) tensor is a *random tensor* $\mathbf{T}$ which we assume to be an element of the tensor-valued *Gauss–Laplace* normal distribution over $\mathbb{R}^{2 \times 2}$ of type independently, identically distributed (i.i.d.) tensor-valued observations.

In the Earth sciences random tensors have only recently been investigated from the statistical point of view. Since the tensors in the Earth sciences are physical quantities and their dimensions are generally low (3 for stress/strain tensors and 6 for elastic material tensors), mathematically approximate/limit distributions of the random eigenspace components are of limited practical value. In fact, the study of random stress/strain has been focused on the following four aspects:

- (1) the exact distribution of the random principal stress/strain components, since the dimension of stress/strain tensors is not greater than three and since the number of measurements is always finite;
- (2) the accuracy of the random eigenspace components. The accuracy is generally not investigated in the mathematical literature of rank-two random tensors. It is, however, a routine indicator that must be attached to any estimated/derived geo-quantity;
- (3) the biases of the random eigenspace components. Since the mapping between a stress/strain tensor and its eigenspace components is non-linear, the random eigenspace components are biased. The biases of the eigenspace components, except for some inequality results on the biases of the random eigenvalues (see, e.g. Cacoullos 1965), have not been well investigated in the mathematical literature on rank-two random tensors. They can have an important role to play in correctly interpreting the estimated stress/strain field geophysically, though and
- (4) the eigendirections.

The eigendirections have been almost always treated as nuisance parameters in nuclear physics and multivariate analysis. Geophysically, the eigendirections are very important and thus cannot be ignored (Xu & Grafarend 1996b).

The first work on the statistical analysis of random tensors in the Earth sciences was to compute the first-order accuracy of the principal eigenvalues of a symmetric, rank-two random tensor (Angelier *et al.* 1982, as an appendix) and probably independently, Soler & van Gelder (1991), Feigl *et al.* (1990). Kagan & Knopoff (1985) studied statistically the first two moments of stochastic 3-D seismic moment tensor invariants, which were used to explain complex fault geometry (Kagan 1992).

On the assumption that a strain tensor or stress tensor has been directly measured or derived from other observations, such a 2-D, symmetric rank-two tensor is a random tensor $\mathbf{T}$. The distribution of the *eigenspace components* of the rank-two random tensor (principal components, principal directions) has been investigated by Xu & Grafarend (1996a,b), which is significantly different from the commonly used *Gauss–Laplace* normal distribution. The possible bias terms of eigenspace components and the non-linear error propagation are also studied in these two papers. By means of the numerical analysis with the power series expansion, the marginal probability density function $f_{\lambda}(\lambda_1, \lambda_2)$ of the random eigenvalues λ_1 and λ_2 has been approximately computed by Cai (2001). Xu (1999a) and Kagan (2000) developed the general distribution of the eigenspace components of the second-order symmetric random tensor, which has been applied to real-life problems in several special cases since an exact distribution theory of eigenspace components can only be constructed under certain special conditions. Specifically, Xu (2002) developed isotropic distribution models for the eigendirections, which allowed the performance of different GPS ambiguity decorrelation methods to be compared (Xu 2001).

These reasons give rise to investigate the statistical inference of eigenspace components of the 2-D symmetric, rank-two random tensor ('random matrix') based upon a linearized multivariate *Gauss–Markov* model which will provide us with the second-order statistics of eigenspace components. The eigenspace synthesis relates the eigenspace elements to the observations by means of a non-linear vector-valued function, thus establishing a special non-linear multivariate *Gauss–Markov* model. For its linearized form, the best linear uniformly unbiased estimation (BLUE) of the eigenspace elements and the best invariant quadratic uniformly unbiased estimate (BIQUE) of its variance–covariance matrix have been successfully constructed. Such a statistical inference of a random matrix is completed by the design of a linear hypothesis test (Cai *et al.* 2005). With these models we could successfully perform the statistical inference of the *eigenspace components* vector and the variance–covariance matrix of the *Gauss–Laplace* normally distributed observations of a random deformation tensor.

While station velocity diagrams demonstrate relative motions among stations, strain rate diagrams show the *in situ* strain concentration rate which is directly connected to local stress concentration rates and possibly also to seismic hazard potentials (Ward 1994). In strain analysis the displacements are considered as continuously differentiable according to the surface coordinates. The strain tensor components determined by means of the positional changes of the observation stations can be used for the computation of the stress tensors' components, taking into account the properties of the available materials within the investigation area. Therefore, the strain analysis can be considered as a basis of a dynamic model whereas the classical deformation analysis is similar to a kinematic model (see, e.g. Flügge 1972; Means 1976; Grafarend 1977; Brunner 1979; Altiner 1999).

The first geodetic deformation strain analysis based on the geodetic horizontal displacement was published by Tsuboi (1932), who computed the strain pattern using the horizontal displacement of control points in the Tango area of Japan during the period 1900 to 1930, which contains the Tango earthquake of magnitude 7.4 in 1927. The classic strain calculation methods from geodetic observations (distance, direction etc.) are contributed to Frank (1966), Savage & Hastie (1966) and Prescott (1976). Until now, more and more papers have been published, dealing with the stress, strain or strain rate deformation on the Earth's surface, such as Angelier (2002), Haines & Holt (1993), Kahle *et al.* (1995), Kreemer *et al.* (2000), Savage *et al.* (2001), Scherneck *et al.* (2002) and Shen *et al.* (1996).

With the benefit of the development of space geodesy and the continuous observations of the permanent networks including the *International GPS Service* (IGS) Network, the *International Laser Ranging Service* (ILRS) Network, the *International VLBI Service for Geodesy and Astrometry* (IVS) Network and the *International DORIS Service* (IDS) Network, and their combination *International Terrestrial Reference Frame* (ITRF) by IERS, we can now derive the strain rate tensor observations and estimate the eigenspace component parameters

of these random tensor samples with our newly developed theory (Cai 2004; Cai *et al.* 2005), which addresses not only the present-day deformation pattern but also their continuous change.

In this paper we begin with a review of the results of statistical inference of eigenspace components of the 2-D symmetric, rank-two random tensor ('random matrix'), following Cai *et al.* (2005), that is, the best linear uniformly unbiased estimation (BLUE) of the eigenspace elements and the best invariant quadratic uniformly unbiased estimate (BIQUE) of its variance–covariance matrix. Then the geodynamic setting of the Earth and especially the selected investigated region—the central Mediterranean and Western Europe will be discussed. Thirdly, the ITRF sites are selected according to the history and quality of the ITRF realization series, and the related incremental velocities of selected ITRF sites are computed. Furthermore the methods of derivation for the 2-D geodetic strain rates are introduced in order to obtain these strain rates from the incremental velocities. In the case study, both BLUE and BIQUE models as well as related hypothesis tests are applied to the eigenspace components of the 2-D strain rate tensor observations in the area of the central Mediterranean and Western Europe, as derived from the ITRF92 to ITRF2000 sequential station positions and velocities in Section 3.2. The interpretation and comparison of these results with the geodynamic feature are performed also in this section. The related *linear hypothesis test* has documented large confidence regions for the eigenspace components, namely *eigenvalues and eigendirections*, based upon real measurement configurations. They lead to the statement *to be cautious* with data of type extension and contraction as well as with the orientation of principal stretches.

2 REVIEW OF STATISTICAL INFERENCE OF THE EIGENSPACE COMPONENTS OF A 2-D, SYMMETRIC RANDOM TENSOR

2.1 The eigenspace analysis versus eigenspace synthesis of a 2-D, symmetric random tensor

Let there be given a symmetric 2-D rank-two random strain tensor $\mathcal{T} \in \mathbb{T}_0^2$ which is represented in a commutative left or right orthonormal base $\{\mathbf{e}^1, \mathbf{e}^2\}$, in short $\mathbf{e}^i \otimes \mathbf{e}^j$ for all $i, j \in \{1, 2\}$. ' $\otimes$ ' denotes the tensor product. Accordingly $\mathbf{T} = [t_{ij}] \in \mathbb{R}^{2 \times 2}$ is called the *matrix representation* of the 2-D symmetric rank-two tensor.

Corollary 1: (i) *Eigenvalue–eigenvector analysis:*

For a symmetric tensor $\mathcal{T} \in \mathbb{R}^{2 \times 2}$ the eigenvalues λ_1 and λ_2 as well as the orientation parameter α , which generate the orthonormal matrix $\mathbf{U} \in \mathbb{R}^{2 \times 2}$ of eigenvectors, are analytically represented by the matrix elements of $\mathbf{T}$ through

$$\begin{aligned} \lambda_1 \in \mathbb{R}, \quad \lambda_1 &= \frac{1}{2} \left(t_{11} + t_{22} + \sqrt{(t_{11} - t_{22})^2 + 4t_{12}^2} \right), \\ \lambda_2 \in \mathbb{R}, \quad \lambda_2 &= \frac{1}{2} \left(t_{11} + t_{22} - \sqrt{(t_{11} - t_{22})^2 + 4t_{12}^2} \right), \\ \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \quad \alpha &= \frac{1}{2} \arctan 2t_{12}/(t_{11} - t_{22}). \end{aligned} \quad (1)$$

(ii) *Eigenvalue–eigenvector synthesis:*

Given the eigenvalues λ_1 and λ_2 as well as the orientation parameter α , which generate the orthonormal matrix $\mathbf{U} \in \mathbb{R}^{2 \times 2}$ of eigencolumns, the symmetric tensor $\mathcal{T} \in \mathbb{R}^{2 \times 2}$ is synthetically represented by the matrix $\mathbf{T}$ with elements

$$\begin{aligned} t_{11} &= \lambda_1 \cos^2 \alpha + \lambda_2 \sin^2 \alpha, \\ t_{12} = t_{21} &= \frac{1}{2} (\lambda_1 - \lambda_2) \sin 2\alpha, \\ t_{22} &= \lambda_1 \sin^2 \alpha + \lambda_2 \cos^2 \alpha. \end{aligned} \quad (2)$$

On the basis of *Corollary 1* (eigenspace analysis and eigenspace synthesis) we are able to portray the symmetric 2D-strain tensor $\mathcal{T}$, which can be visualized as *strain ellipse*, if $\text{sign } \lambda_1 = \text{sign } \lambda_2$, and as *strain hyperbola* if $\text{sign } \lambda_1 \neq \text{sign } \lambda_2$ in Figs 1 and 2, respectively.

2.2 The linearized multivariate Gauss–Markov model for the estimation of eigenspace components of a symmetric rank-two random tensor

The *Corollary 1* has documented that the eigenspace synthesis of a symmetric random tensor is non-linear in terms of the tensor-valued observations and there is no simple probability density function of the distribution of random eigenspace components. Accordingly, it is unable to derive the exact sampling distribution of the eigenvalues λ_1 and λ_2 as well as the orientation parameter α directly (Cai 2001, 2004). As a solution, we have derived the linearized counterpart for sampling the eigenspace synthesis parameters from the originally non-linear observation equations. The Σ -BLUE of eigenspace components and their variance–covariance component matrix estimate of type BIQUE has been developed in accordance with the formulas presented earlier by Cai *et al.* (2001), and completed by Cai (2004) and Cai *et al.* (2005), which are summarized herewith.

$$\mathbf{T} = [t_{ij}] \in \mathbb{R}^{2 \times 2}, \quad \text{vech } \mathbf{T} = \begin{bmatrix} t_{11} \\ t_{21} \\ t_{22} \end{bmatrix} =: \mathbf{y} \in \mathbb{R}^{3 \times 1} \text{ (read : vec half),}$$

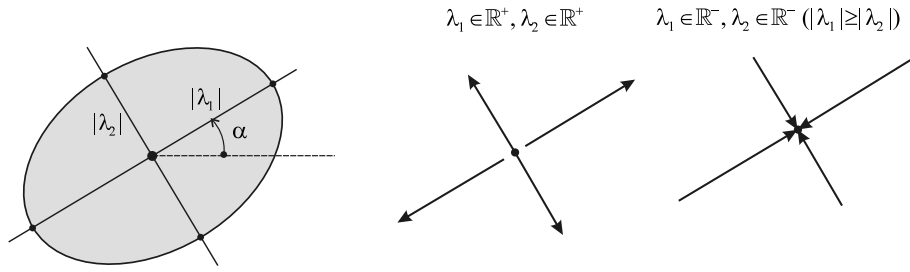


Figure 1. Strain ellipse, sign $\lambda_1 = \text{sign } \lambda_2$, extension or contraction.

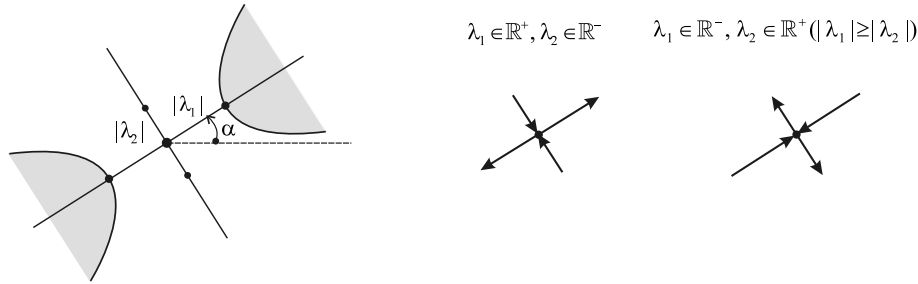


Figure 2. Strain hyperbola, sign $\lambda_1 \neq \text{sign } \lambda_2$, extension in one direction, contraction in the other direction or vice versa.

Suppose that n samples of $\mathbf{T}$, namely $\mathbf{T}_1, \mathbf{T}_2, \dots, \mathbf{T}_n$ have been observed, whose related vectorized forms are $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$. Here we design an array of vectorized tensor coordinates $y_1 := t_{11}, y_2 := t_{21}, y_3 := t_{22}$, indexed to the number of the sample. Then

$$\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n] = \begin{bmatrix} y_{1.1} & \cdots & y_{1.n} \\ y_{2.1} & \cdots & y_{2.n} \\ y_{3.1} & \cdots & y_{3.n} \end{bmatrix}, \quad \mathbf{Y} \in \mathbb{R}^{3 \times n}, \quad (3)$$

has a variance–covariance matrix that follows when $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are i.i.d. 3×1 random vectors, each with same variance–covariance matrix Σ_y , as

$$D\{\text{vec} \mathbf{Y}\} = \begin{bmatrix} \Sigma_y & 0 & \cdots & 0 \\ 0 & \Sigma_y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Sigma_y \end{bmatrix} = \mathbf{I}_n \otimes \Sigma_y, \quad D\{\text{vec} \mathbf{Y}\} \in \mathbb{R}^{3n \times 3n}, \quad (4)$$

where $\otimes$ now denotes the *Kronecker–Zehfuss* product of matrices (see Grafarend & Schaffrin 1993).

Using the eigenspace analysis *versus* eigenspace synthesis as in *Corollary 1*, we can define the non-linear *Gauss–Markov* model which is presented by (5), (6) and (4).

Non-linear Model: Special non-linear multivariate Gauss–Markov model for sampling the eigenspace synthesis:

$$\mathbf{Y} = \mathbf{F}(\xi) \mathbf{1}' + \tilde{\mathbf{E}}, \quad (5)$$

$$\text{1st moments: } E\{\mathbf{Y}\} = E\left\{ \begin{bmatrix} y_{1.1} & \cdots & y_{1.n} \\ y_{2.1} & \cdots & y_{2.n} \\ y_{3.1} & \cdots & y_{3.n} \end{bmatrix} \right\} = \mathbf{F} \mathbf{1}',$$

where $\mathbf{1}$ denotes the $n \times 1$ ‘summation vector’ with all its entries being 1.

$$\mathbf{F} := \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \cos^2 \alpha + \lambda_2 \sin^2 \alpha \\ \frac{1}{2}(\lambda_1 - \lambda_2) \sin 2\alpha \\ \lambda_1 \sin^2 \alpha + \lambda_2 \cos^2 \alpha \end{bmatrix} = \begin{bmatrix} \xi_1 \cos^2 \xi_3 + \xi_2 \sin^2 \xi_3 \\ \frac{1}{2}(\xi_1 - \xi_2) \sin 2\xi_3 \\ \xi_1 \sin^2 \xi_3 + \xi_2 \cos^2 \xi_3 \end{bmatrix}, \quad (6)$$

$$\left. \begin{array}{l} \xi_1 := \lambda_1 \\ \xi_2 := \lambda_2 \\ \xi_3 := \alpha \end{array} \right\} \text{ and } \left\{ \begin{array}{l} f_1 := t_{11} \\ f_2 := t_{21} \\ f_3 := t_{22} \end{array} \right.$$

2nd moments (independent between (but not among) observations with identical distributions):

$$D\{\text{vec}\mathbf{Y}\} = \mathbf{I}_n \otimes \Sigma_y, \quad (4)$$

$$D\{\text{vec}\mathbf{Y}\} = \Sigma \in \mathbb{R}^{3n \times 3n}, \quad \Sigma \text{ positive-definite, } \text{rk } \Sigma = 3n,$$

ξ , $E\{\mathbf{Y}\}$, $\mathbf{Y} - E\{\mathbf{Y}\} = \tilde{\mathbf{E}}$ unknown, Σ unknown (but patterned).

In order to estimate the eigenspace components of a symmetric random tensor of second order, the non-linear observation equations will be linearized. The linearization of the non-linear observation eqs (5) is now presented in detail: **Linearization of non – linear observation equations:**

$$\text{First set of observations: } \begin{bmatrix} y_{1,1} \\ y_{2,1} \\ y_{3,1} \end{bmatrix} = \begin{bmatrix} t_{11,1} \\ t_{21,1} \\ t_{22,1} \end{bmatrix}.$$

$$\text{Define : } \xi_0 := \begin{bmatrix} \lambda_{1,1} \\ \lambda_{2,1} \\ \alpha_{,1} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} t_{11,1} + t_{22,1} + \sqrt{(t_{11,1} - t_{22,1})^2 + 4t_{21,1}^2} \\ t_{11,1} + t_{22,1} - \sqrt{(t_{11,1} - t_{22,1})^2 + 4t_{21,1}^2} \\ \arctan 2t_{21,1}/(t_{11,1} - t_{22,1}) \end{bmatrix}, \quad \Delta\xi := \xi - \xi_0, \quad (7)$$

$$\text{'Linearized non – linear model : } \mathbf{F}(\xi) - \mathbf{F}(\xi_0) = \mathbf{J}(\xi_0)\Delta\xi + \mathcal{O}(\Delta\xi \otimes \Delta\xi), \quad (8)$$

where the ‘Jacobi matrix’ will be

$$\mathbf{A} := \mathbf{J}(\xi_0) = \left[\begin{array}{ccc} \frac{\partial f_1}{\partial \lambda_1} & \frac{\partial f_1}{\partial \lambda_2} & \frac{\partial f_1}{\partial \alpha} \\ \frac{\partial f_2}{\partial \lambda_1} & \frac{\partial f_2}{\partial \lambda_2} & \frac{\partial f_2}{\partial \alpha} \\ \frac{\partial f_3}{\partial \lambda_1} & \frac{\partial f_3}{\partial \lambda_2} & \frac{\partial f_3}{\partial \alpha} \end{array} \right]_{\xi=\xi_0} = \left[\begin{array}{ccc} \cos^2 \alpha_{,1} & \sin^2 \alpha_{,1} & (\lambda_{2,1} - \lambda_{1,1}) \sin 2\alpha_{,1} \\ \frac{1}{2} \sin 2\alpha_{,1} & -\frac{1}{2} \sin 2\alpha_{,1} & -(\lambda_{2,1} - \lambda_{1,1}) \cos 2\alpha_{,1} \\ \sin^2 \alpha_{,1} & \cos^2 \alpha_{,1} & -(\lambda_{2,1} - \lambda_{1,1}) \sin 2\alpha_{,1} \end{array} \right]. \quad (9)$$

The special linearized multivariate *Gauss–Markov model* for sampling the eigenspace of a symmetric random matrix is summarized by (10)–(15).

Linearized Model: Special linearized multivariate *Gauss–Markov model* for sampling the eigenspace synthesis:

$$\mathbf{Y} = \mathbf{F}(\xi_0)\mathbf{1}' + [\mathbf{A}(\xi - \xi_0)]\mathbf{1}' + \mathbf{E}, \quad (10)$$

$$\text{'vectorized version': } \text{vec}\mathbf{Y} = \mathbf{1} \otimes \mathbf{F}(\xi_0) + (\mathbf{1} \otimes \mathbf{A})(\xi - \xi_0) + \text{vec}\mathbf{E}, \quad (11)$$

$$\text{new notations: } \text{vec}\mathbf{Y}_0 = \mathbf{1} \otimes \mathbf{F}(\xi_0), \quad \mathbf{A} = (\mathbf{1} \otimes \mathbf{A}),$$

$$\text{'1st moments': } \mathbf{A}(\xi - \xi_0) + \text{vec}\mathbf{Y}_0 = E\{\text{vec}\mathbf{Y}\}, \quad \text{vec}\mathbf{Y} \in \mathbb{R}^{3n \times 1}, \quad (12)$$

$$\text{'2nd moments': } D\{\text{vec}\mathbf{Y}\} = \mathbf{I}_n \otimes \Sigma_y =: \Sigma \in \mathbb{R}^{3n \times 3n}, \quad \Sigma_y \text{ positive - definite, } \text{rk } \Sigma = 3n; \quad (13)$$

$$\Sigma, E\{\mathbf{Y}\}, \mathbf{Y} - E\{\mathbf{Y}\} + \mathcal{O}[(\xi - \xi_0) \otimes (\xi - \xi_0)]\mathbf{1}'_n = \mathbf{E} \text{ unknown, } \xi \text{ unknown.}$$

With these definitions and the observations of a random tensor we can *first* estimate the eigenspace components ξ of type Σ -BLUE (Best Linear Uniformly Unbiased Estimation) which are collected in

Theorem 2: $\hat{\xi}$ Σ —BLUE of ξ , the eigenspace components of a symmetric random tensor: The Σ -BLUE $\hat{\xi}$ of ξ in the special linearized multivariate *Gauss–Markov Model* is

$$\hat{\xi} = \xi_0 + \Delta\hat{\xi} \quad \text{with}$$

$$\begin{aligned} \Delta\hat{\xi} &= (\hat{\xi} - \xi_0) = \mathbf{L}(\text{vec}\mathbf{Y} - \text{vec}\mathbf{Y}_0) \\ &= (\mathbf{A}'\Sigma^{-1}\mathbf{A})^{-1}\mathbf{A}'\Sigma^{-1}(\text{vec}\mathbf{Y} - \text{vec}\mathbf{Y}_0) \\ &= \left[\frac{1}{n}\mathbf{1}' \otimes (\mathbf{A}'\Sigma_y^{-1}\mathbf{A})^{-1}\mathbf{A}'\Sigma_y^{-1} \right] (\text{vec}\mathbf{Y} - \text{vec}\mathbf{Y}_0) \\ &= \frac{1}{n}(\mathbf{A}'\Sigma_y^{-1}\mathbf{A})^{-1}\mathbf{A}'\Sigma_y^{-1}(\mathbf{Y} - \mathbf{Y}_0)\mathbf{1} \end{aligned} \quad (14)$$

with the related dispersion matrix

$$D\{\hat{\xi}\} = D\{\Delta\hat{\xi}\} =: \Sigma_{\hat{\xi}} = \frac{1}{n}(\mathbf{A}'\Sigma_y^{-1}\mathbf{A})^{-1} = \frac{1}{n}(\mathbf{A}^{-1}\Sigma_y^{-1}\mathbf{A}^{-1})'. \quad (15)$$

Since the variance–covariance component matrix Σ_y of the observation vector $\text{vec } \mathbf{Y}$ is unknown we have to estimate such a dispersion matrix empirically. $\hat{\xi}$ as the BIQUUE (Best Invariant Quadratic Uniformly Unbiased Estimate) of Σ_y is summarized in *Theorem 3*, and for example, proven in Grafarend & Schaffrin (1993).

Theorem 3: The sample variance–covariance component matrix $\hat{\Sigma}_y$ of type BIQUUE of a symmetric random tensor:

The sample variance–covariance component matrix $\hat{\xi}$ of type BIQUUE for the vectorized observations of a symmetric random tensor of second order is

$$\hat{\Sigma}_y = \frac{1}{n-1}(\mathbf{Y} - \mathbf{Y}_0)\left(\mathbf{I}_n - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)(\mathbf{Y} - \mathbf{Y}_0)' =: \frac{1}{n-1}(\mathbf{A}\mathbf{\Omega}\mathbf{A}'), \text{ or } \mathbf{\Omega} = n(n-1)\hat{\Sigma}_{\hat{\xi}}. \quad (16)$$

2.3 Testing eigenspace components of a symmetric random tensor of second order

In order for the estimated tensors to be significant, statistical inference has to be performed, based on the three elements of validation, namely sampling distribution, parameter estimation (point and interval estimates), as well as hypothesis testing. With respect to the previous eigenspace analysis in terms of a special linearized *multivariate Gauss–Markov model* providing $\Delta\hat{\xi}$ as BLUE of $\Delta\xi$ and $\hat{\Sigma}_y$ as BIQUUE of Σ_y , we are able to start the *statistical validation of the eigenspace components* and their *variance–covariance matrix* assuming a *Gauss–Laplace normal distribution* of the observed (or derived) deformation tensor elements. In detail, with the estimated eigenelements of a random strain rate tensor and their dispersion matrix, the following *multivariate test strategy* will be performed where ξ denotes the 3×1 vector of eigenelements:

- (1) Test for the eigenspace parameter vector $\xi = \xi_0$ with Σ_y unspecified (see Box 2);
- (2) Test for a distinct element of the eigenspace parameter vector with *Student t-test* (see Box 3);
- (3) Test for the variance–covariance matrix $\Sigma_y = \Sigma_0$ (see Box 4);
- (4) Test for the eigenspace parameter vector and variance–covariance matrix $\xi = \xi_0, \Sigma_y = \Sigma_0$ (see Box 5);
- (5) *Eigen inference* about the orthonormal transformed parameters η ;
- (6) The general linear hypothesis test with the growth curve model for eigenspace parameters.

More on multivariate hypothesis testing can be found in Wilks (1932), Lawley (1938) and Hotelling (1951), Kendall & Stuart (1958), Anderson (1958, 1984), James (1964), Giri (1977), Srivastava & Khatri (1979), Muirhead (1982), Siotani *et al.* (1985), Rencher (1998), Koch (1999), Srivastava (2002) and Cai (2004).

With a case study, both model and hypothesis tests will later be applied to the observations of a random strain rate tensor as explained in Section 3.

3 THE ANALYSIS OF THE EIGENSPACE COMPONENTS OF THE STRAIN RATE TENSOR IN CENTRAL MEDITERRANEAN AND WESTERN EUROPE, 1992–2000

3.1 Geodynamic setting of the investigated region

As pointed out by Grafarend & Voosoghi (2003), the European and Mediterranean area, is known as an extraordinary natural laboratory for the study of seismotectonic processes. This area is geologically, geophysically, and geodetically, one of the best-studied regions on the Earth's surface. The research interest encompasses the past 100 yr and consequently a huge number of publications exists addressing local and regional geodynamical processes. A list of sample references are also provided in some papers, such as, McKenzie (1970), Cross *et al.* (1987), Jackson & McKenzie (1988), Argus *et al.* (1989), Smith *et al.* (1990), Castellarin *et al.* (1992), Mueller *et al.* (1992), James & Lambert (1993), Ward (1994), Reilinger *et al.* (1997a,b), Clarke *et al.* (1998), Kahle *et al.* (1998), DeMets & Dixon (1999) and Caporali *et al.* (2001). More recently, the present-day crustal motions in the central Mediterranean area and in Western Europe are studied by Anzidei *et al.* (2001), Devoti *et al.* (2002a,b), Grenczy (2002), Caporali *et al.* (2003b), Jiménez-Munt *et al.* (2003) and Nocquet & Calais (2003) with the newly developed continuous observation data from space geodetic networks, such as permanent GPS networks.

The recent major tectonic processes occur within the large-scale kinematic framework of active seafloor spreading (divergence) in the Atlantic Ocean and the African–Eurasian convergence (subduction) boundary in the Mediterranean. The spreading rate in the South Atlantic ($\sim 40 \text{ mm yr}^{-1}$), higher than that of the North Atlantic ($\sim 20 \text{ mm yr}^{-1}$) leads to a gradual counter-clockwise rotation of the African plate, resulting in a NNW-directed push against Eurasia, which in turn leads to a lithospheric shortening of $5\text{--}6 \text{ mm yr}^{-1}$, increasing to 40 mm yr^{-1} in active subduction zones (Argus *et al.* 1989). With NW–SE-oriented spreading in the North Atlantic, the whole region is expected to be under compression, particularly the Mediterranean area. It is widely recognized that the Mediterranean area represents the collision zone between the African/Arabian and Eurasian plates, but the deformation pattern of this region is characterized by a complex space-time distribution of compressional and tensional events. The active volcanoes and earthquakes in this region show evidently a high seismic activity, due to relatively strong tectonic forces that govern the compression zone. Moreover, a widespread intraplate seismicity occurs in the region which illustrates that the plate collision zone is complex and not sharply defined.

Therefore, we would choose the Western European and Mediterranean areas to perform the statistical analysis of the eigenspace components of 2-D strain rate tensor observations.

The European and Mediterranean area can be divided into three main subregions with distinct geodynamic features, namely Western Europe, Northern Europe and the Alpine–Mediterranean subregions. Based on the space geodetic observations history (10 or more years) we will focus on the behaviour of significant active deformation in the North of the Western Mediterranean with respect to Europe, that is,

two of the subregions, Alpine–Mediterranean and Western Europe. Within Western Europe, weak seismic activity is observed. The area is characterized as a field of compressional tectonics. A generalized stress map of Europe (Mueller *et al.* 1992) indicates a generally NW–SE uniform orientation for the maximum compressive horizontal principal stress in Western Europe. The Alpine–Mediterranean region marks a broad transformation zone between the African, Arabian and Eurasian plates. The region is expected to be largely under compression. It is characterized as a region of intensive seismic activity. The tectonic evolution of this region is strongly affected by the convergence of the microplates (Voosoghi 2000). In order to get reliable observations on the ongoing tectonic processes in this area, we should select the sites (see Section 3.2) that are not affected by local geophysical phenomena.

3.2 The selection of ITRF Sites and data preparation

Before we begin to choose from the ITRF series results and the stations in the studied regions, let us review the history of IERS and the realization of ITRF.

3.2.1 The realization of ITRF

The IERS was established as the International Earth Rotation Service in 1987 by the International Astronomical Union and the International Union of Geodesy and Geophysics, and it began operation on 1 January 1988. In 2003 it was renamed to *International Earth Rotation and Reference Systems Service* (IERS 2003). The primary objectives of the IERS are to serve the astronomical, geodetic and geophysical communities by providing the International Celestial Reference System (ICRS) and its realization, the International Celestial Reference Frame (ICRF); the International Terrestrial Reference System (ITRS) and its realization, the International Terrestrial Reference Frame (ITRF); Earth orientation parameters required to study Earth orientation variations and to transform the ICRF into the ITRF, and vice versa; geophysical data to interpret time/space variations in the ICRF, ITRF, or Earth orientation parameters, and model such variations and the standards, constants, and conventions encouraging international adherence.

Until now, ten successive realizations of the ITRF have been published, starting with ITRF88 and ending with ITRF2000, each of which superseded its predecessor.

The ITRF network has improved with time in terms of the number of sites and collocations as well as their distribution over the globe. The ITRF88 network, has about 100 sites and 22 collocations (VLBI/SLR/LLR), and the ITRF2000 network contains about 500 sites and 101 collocations (VLBI/SLR/GPS/DORIS). With the improvements of the analysis strategy by the IERS Analysis Centers and the ITRF combination, the ITRF position and velocity precisions have also improved with time. As the current Reference Realization of the ITRS, the ITRF2000 is intended to be a standard solution for geo-referencing and all Earth science applications (Altamimi *et al.* 2002).

The construction of ITRF is based on the combination of sets of station coordinates (SSCs) and velocities derived from observations of space geodetic techniques, such as VLBI, SLR, GPS, LLR and DORIS by various analysis centres. For the determination of a station's position in one of the ITRF, the station is assigned to a specific tectonic plate. The point position of the station at time, t , on the surface of the solid earth, is expressed as (Boucher & Altamimi 1993):

$$\mathbf{X}(t) = \mathbf{X}_0 + \mathbf{V}_0(t - t_0) + \sum_i \Delta \mathbf{X}_i(t), \quad (17)$$

where

$\Delta \mathbf{X}_i$: corrections to the various time-changing effects;

$\mathbf{X}_0$: position at epoch t_0 ;

$\mathbf{V}_0$: velocity at epoch t_0 ;

t_0 : initial reference epoch (i.e. 1988.0).

The coordinates of sites on the Earth's surface slowly change (by up to 10 cm per year, or so) due to the motion of the tectonic plates—a component which is familiarly known as 'continental drift'. The velocity $\mathbf{V}_0$ should be expressed as

$$\mathbf{V}_0 = \mathbf{V}_{\text{plate}} + \mathbf{V}_{\text{ice}} + \mathbf{V}_r, \quad (18)$$

where

$\mathbf{V}_{\text{plate}}$: is the horizontal velocity due to plate tectonic motion, which can be described by a geophysical and geological angular velocity vector ω_{YY} (a Cartesian rotation vector with components $\omega_X, \omega_Y, \omega_Z$) of the absolute plate motion models, such as the more recent NNR-NUVEL-1A;

$\mathbf{V}_{\text{ice}}$: is the vertical velocity due to post glacial rebound, to be computed from models such as ICE-4G (Peltier 1995);

$\mathbf{V}_r$: is the incremental velocity.

In the data analysis $\mathbf{X}_0$, and $\mathbf{V}_r$ should be estimated as parameters. When adjusting parameters, in particular velocities, the IERS orientation should be kept at all epochs, which means to ensure the alignment at a reference epoch and the time evolution through a no net rotation condition with regards to horizontal tectonic motion over the whole Earth.

3.2.2 The selection of ITRF series results and the stations

With the review of the history and the realization of IERS and the realization of ITRF we will further discuss the selection of ITRF series results and the stations in our studied region.

In order to fulfil the quality requirements (in determination of successful strain calculation in Western Europe and central Mediterranean in our study), the ITRF sites are selected with high geodetic quality, satisfying the following criteria:

- (i) it is a primary permanent station after the standards of IERS since 1991 or collocated with several observational techniques
- (ii) it has continuous observation for at least 3 yr in 1992;
- (iii) the station coordinate accuracy better than ± 8 mm;
- (iv) and the velocity accuracy better than ± 3 mm yr⁻¹.

Therefore, for our study we have chosen just eight primary stations and one secondary station (Noto) from ITRF92 and the following series with all of these collocating VLBI and SLR techniques. They are located in Western Europe and the central Mediterranean and are listed in Table 1.

3.2.3 The computation of incremental velocities of ITRF stations

Since we need the incremental velocities in determination of the strain rate, the published ITRF velocity $\mathbf{V}_0$ should be converted to incremental velocities $\mathbf{V}_r$, with respect to, for example, the Eurasian fixed plate by subtracting the rigid motion of Eurasia which is computed by the angular velocity vector ω_{YY} of the Eurasian plate in the absolute plate motion models of every ITRF realizations:

$$\mathbf{V}_r = \mathbf{V}_0 - \mathbf{V}_{\text{plate}} = \mathbf{V}_0 - \omega_{YY} \times \mathbf{X} = \begin{bmatrix} V_{X0} \\ V_{Y0} \\ V_{Z0} \end{bmatrix} - \begin{bmatrix} 0 & Z & -Y \\ -Z & 0 & X \\ Y & -X & 0 \end{bmatrix} \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix}. \quad (19)$$

It should be noted that although the station Noto is located on the African plate, the motion of Matera was recognized as neither purely Euro-Asiatic nor African. For the calculation of the strain rate filed in the selected region, we have to compute the incremental velocity with respect to the same rigid plate, for example, the Eurasian plate. Table 2 summarizes the component values of ω_{YY} related with every ITRF realization (Boucher & Altamimi 2001).

In the 2-D case we need the surface (horizontal) incremental velocities. Since the incremental velocity from eq. (19) is relatively small, they can approximately be transformed from the global geocentric Cartesian coordinate system to the local geodetic system (Seeber 2003),

Table 1. Catalogue of selected IERS Sites based on ITRF92 in Europe.

Domes NB.	Site name	Tech.	ID	Country	Long.		Lat.		Plate	(*)
					d	m	d	m		
10002S001	Grasse	SLR	7835	France	6	55	43	45	Eura	P C
10402S002	Onsala	VLBI	7213	Sweden	11	55	57	24	Eura	P C
11001S002	Graz	SLR	7839	Austria	15	30	47	04	Eura	P C
12711S001	Bologna	VLBI	7230	Italy	11	21	44	29	Eura	P C
12717S001	Noto	VLBI	7547	Italy	14	59	36	53	Afr/Eura	S C
12734S001	Matera	SLR	7939	Italy	16	37	40	42	Eura	P C
13407S010	Madrid	VLBI	1565	Spain	355	44	40	26	Eura	P C
14001S001	Zimmerwald	SLR	7810	Switzerland	7	28	46	53	Eura	P C
14201S004	Wetzell	VLBI	7224	Germany	12	53	49	09	Eura	P C

(*) P: Primary S: Secondary C: Collocation.

Table 2. The estimation of ω_{YY} .

ITRF	ω_X (mas yr ⁻¹)	ω_Y (mas yr ⁻¹)	ω_Z (mas yr ⁻¹)
92	0.21	0.52	-0.68
93	0.32	0.78	-0.67
94	0.20	0.50	-0.65
96	0.20	0.50	-0.65
97	0.20	0.50	-0.65
2000	0.081	0.490	-0.792

Table 3. Horizontal station velocities.

Site	Velocity (mm yr ⁻¹)		Incremental velocity (mm yr ⁻¹)		Vlocity (mm yr ⁻¹)		Incremental velocity (mm yr ⁻¹)		Velocity (mm yr ⁻¹)		Incremental velocity (mm yr ⁻¹)	
	East	North	East	North	East	North	East	North	East	North	East	North
ITRF92					ITRF93				ITRF94			
Grasse	20.52	12.05	−0.49	−2.99	25.77	19.99	2.02	−2.73	20.72	12.76	0.62	−1.63
Onsala	16.12	15.20	−3.39	0.97	21.05	23.11	−2.44	1.60	16.95	13.88	−1.69	0.26
Graz	22.04	16.37	−0.02	2.76	28.75	22.78	2.97	2.21	22.15	14.88	1.06	1.86
Bologna	22.23	17.23	0.49	2.95	26.37	25.21	1.41	3.62	22.38	16.14	1.60	2.47
Noto	20.91	21.91	−2.16	8.20	25.62	29.06	−0.40	8.35	20.88	19.39	−1.19	6.27
Matera	22.62	19.37	−0.40	5.98	28.78	26.88	2.41	6.65	24.44	16.92	2.43	4.10
Madrid	19.71	16.82	0.23	0.44	21.09	25.49	0.09	0.73	18.68	16.12	0.06	0.44
Zimmerwald	21.66	16.90	1.04	1.95	25.94	25.15	2.35	2.56	19.70	16.32	−0.02	2.01
Wettzell	19.46	15.77	−1.81	1.69	24.24	23.41	−0.65	2.14	19.66	14.16	−0.68	0.69
ITRF96					ITRF97				ITRF2000			
Grasse	20.21	14.32	0.11	−0.07	19.79	13.06	−0.31	−1.33	20.34	14.71	−0.35	−0.01
Osala	17.34	13.11	−1.31	−0.51	17.19	13.18	−1.45	−0.44	17.25	13.59	−0.66	−0.69
Graz	22.13	14.29	1.04	1.27	21.89	13.46	0.80	0.43	22.14	14.46	0.72	0.55
Bologna	23.83	15.14	3.04	1.46	23.12	15.32	2.34	1.64	23.36	16.14	2.03	1.83
Noto	22.38	19.09	0.30	5.97	21.96	17.73	−0.11	4.61	21.28	18.05	−2.12	4.07
Matera	23.82	18.52	1.81	5.70	23.53	17.36	1.52	4.55	23.70	18.09	0.71	4.32
Madrid	19.65	15.05	1.02	−0.63	19.13	14.62	0.51	1.06	18.98	15.68	−0.58	0.40
Zimmerwald	18.60	15.01	−1.13	0.70	19.04	13.96	−0.68	−0.35	20.14	15.07	0.13	0.39
Wettzell	20.51	13.37	0.17	−0.10	20.19	13.41	−0.15	−0.06	20.27	14.37	−0.17	0.18

using

$$\begin{bmatrix} V_{Er} \\ V_{Nr} \\ V_{Ur} \end{bmatrix} = \begin{bmatrix} -\sin L & \cos L & 0 \\ -\sin B \cos L & -\sin B \sin L & \cos B \\ \cos B \cos L & \cos B \sin L & \sin B \end{bmatrix} \begin{bmatrix} V_{Xr} \\ V_{Yr} \\ V_{Zr} \end{bmatrix}, \quad (20)$$

where (B, L) are the geodetic ellipsoidal coordinates of a particular ITRF site. They are converted from the global Cartesian coordinates of the ITRF sites (X, Y, Z) with respect to the GRS80 reference ellipsoid. The local geodetic system is defined as follows (Vanicek & Krakiwsky 1986): It is topocentric (T); the U -axis is the outward ellipsoid normal passing through T; the E -axis is directed towards *geodetic East*; the N -axis is directed towards *geodetic North*; Therefore, 2-D surface (horizontal) velocities are the first two elements of $(V_E V_N)$.

Based on the procedures (19) and (20) derived above we can compute the horizontal incremental velocities of every ITRF realizations with respect to the Eurasian ‘fixed’ plate motion model (see Table 3). They are listed in Table 3 together with the ITRF92 to ITRF2000 velocity solutions of the selected stations in the central Mediterranean and in Western Europe. These incremental velocities will be used to compute the strain rates in the next section. Therefore, together with the principal strain rates for every epoch they are also illustrated in the next section.

3.3 The computation of the 2-D geodetic strain rate tensor

The main objective of this study is to analyse the eigenspace component parameters of the 2-D strain rate tensor, which can be derived from the 2-D horizontal incremental velocities of every ITRF series realizations (ITRF92 to ITRF 2000) on the selected sites in the central Mediterranean and Western Europe with the methods introduced in the following.

When we select geodetic sites as vertices of convex polygons we can evaluate the strain tensor of the polygon by using the horizontal velocities. Let $[V_{E_i} \ V_{N_i}]$ be the known horizontal incremental velocity vector (20) of the polygon vertex ‘ i ’ along the East and North directions in the local geodetic coordinate system; the following approximation can be written (Devoti *et al.* 2002a)

$$\begin{bmatrix} V_{E_i} \\ V_{N_i} \end{bmatrix} = \begin{bmatrix} V_{E_B} \\ V_{N_B} \end{bmatrix} + \begin{bmatrix} \frac{\partial V_E}{\partial E} & \frac{\partial V_E}{\partial N} \\ \frac{\partial V_N}{\partial E} & \frac{\partial V_N}{\partial N} \end{bmatrix} \begin{bmatrix} \Delta E_i \\ \Delta N_i \end{bmatrix}, \quad (21)$$

$$\Leftrightarrow \mathbf{V}_{EN_i} = \mathbf{V}_{EN_B} + \mathbf{L} \Delta \mathbf{X}_{EN}$$

where $[V_{E_B} \ V_{N_B}]$ is the unknown velocity vector for a reference internal point ‘ B ’, $\mathbf{L}$ the velocity gradient tensor, $[\Delta E_i \ \Delta N_i]$ the coordinate difference between the site ‘ i ’ and the reference point ‘ B ’, computed, as parallel and meridian arc length, respectively. For the infinitesimal strain rate we have the strain rate tensor $\mathbf{T} = (\mathbf{L} + \mathbf{L}^T)/2$ and the rotation rate tensors $\mathbf{R} = (\mathbf{L} - \mathbf{L}^T)/2$ (Dermanis & Liveratos 1983).

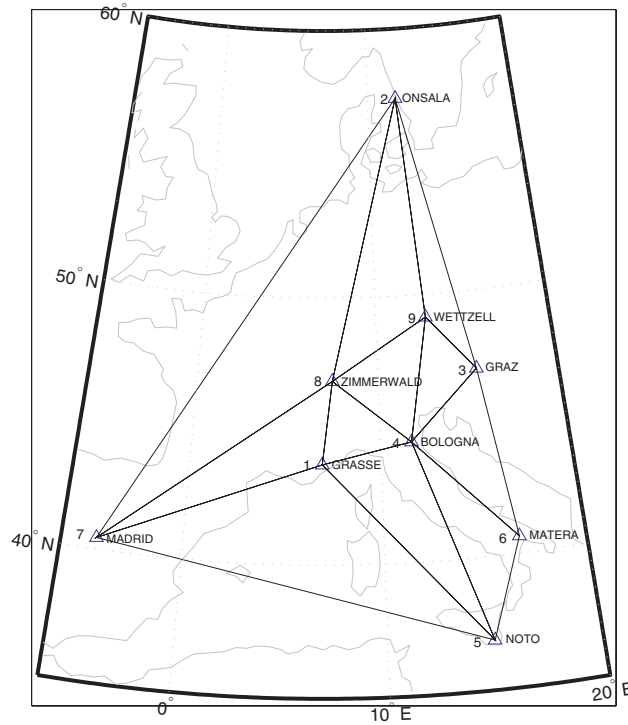


Figure 3. The Delaunay-triangulation of the selected ITRF sites.

The approximation assumes a linear variation of the velocity components with respect to their coordinate differences. This holds true as long as the polygons are properly chosen, not only in terms of area, but also of expected tectonic behaviour. The constant space gradients assumption is just a first order approximation of the underlying tectonic setting. Savage *et al.* (2001) give the formulation for estimating strain and rotation rates in a spherical coordinate system. The spherical solution gives insignificantly different results compared to the Cartesian approximation (21) for networks, for instance from a triangulation, which is several hundred kilometres in aperture.

Since the continuous of velocity field $\mathbf{V}_r$ is unknown, but only discrete values at points 'i' are known, we have to use an interpolation method to obtain the velocity field $\mathbf{V}_r$ at any other point. There exist many interpolation methods, among them: (a) *Finite Element Method (FEM)*- Linear interpolation within each triangle (Grafarend 1986; Dermanis & Grafarend 1993; Straub 1996); (b) *Interpolation using basis functions* (e.g. Haines & Holt 1993) and (c) *Collocation* (least-squares adjustment with spatial prior information) (e.g. Straub & Kahle 1997). Here we shall apply the Finite Element Method to do the linear interpolation within each triangle which is optimally generated by the *Delaunay-triangulation method* among our selected nine stations. The characteristics of Delaunay triangulation are: (1) no triangle side is cut by another; and (2) no points are contained in any other triangle's circumscribed circle. The Delaunay-triangulation of our selected ITRF site is plotted in Fig. 3.

For every triangle we select the centroid as the reference point, from which it is very easy to compute the velocity gradient tensor at the centroid in a very straightforward way; in fact, dealing with three velocity vectors, the problem is solved by inverting a system of linear equations with six unknowns (four tensor components plus two velocity components):

$$\begin{bmatrix} V_{E_1} \\ V_{N_1} \\ V_{E_2} \\ V_{N_2} \\ V_{E_3} \\ V_{N_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta E_1 & \Delta N_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta E_1 & \Delta N_1 \\ 1 & 0 & \Delta E_2 & \Delta N_2 & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta E_2 & \Delta N_2 \\ 1 & 0 & \Delta E_3 & \Delta N_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta E_3 & \Delta N_3 \end{bmatrix} \begin{bmatrix} V_{E_B} \\ V_{N_B} \\ \frac{\partial V_E}{\partial E} \\ \frac{\partial V_E}{\partial N} \\ \frac{\partial V_N}{\partial E} \\ \frac{\partial V_N}{\partial N} \end{bmatrix}. \quad (22)$$

This approach was first proposed by Terada & Miyabe (1929). With the velocity gradient tensor we can derive the 2-D symmetric strain rate tensor $\mathbf{T}$ and the antisymmetric rotation rate tensor $\mathbf{R}$ at the centroid of the discussed Delaunay-triangle network.

$$\mathbf{T} = \frac{1}{2}(\mathbf{L} + \mathbf{L}') = \begin{bmatrix} \frac{\partial V_E}{\partial E} & \frac{1}{2}\left(\frac{\partial V_E}{\partial N} + \frac{\partial V_N}{\partial E}\right) \\ \frac{1}{2}\left(\frac{\partial V_N}{\partial E} + \frac{\partial V_E}{\partial N}\right) & \frac{\partial V_N}{\partial N} \end{bmatrix}, \quad (23)$$

Table 4. Strain rate tensor components, eigenspace components, max. shear strain rates and second invariant.

Epochs	Strain rate tensor components (nanostrain yr ⁻¹)			Eigenspace components (nanostrain yr ⁻¹)(degree)			Max. shear strain rate (nanostrain yr ⁻¹)	Second invariant (nanostrain yr ⁻¹)
	t_{11}	t_{12}	t_{22}	ε_1	ε_2	α_1	$\varepsilon_1 - \varepsilon_2$	$\sqrt{\varepsilon_1^2 + \varepsilon_2^2}$
Triangle 4 - 5 - 6 (Bologna–Noto - Matera) (No. 10)								
ITRF92	1.6621	2.4987	-5.7591	2.4250	-6.5220	16.9781	8.9470	6.9582
ITRF93	6.9357	3.3247	-4.8325	7.8100	-5.7068	14.7339	13.5168	9.6728
ITRF94	8.1550	2.4147	-4.8161	8.5900	-5.2511	10.2108	13.8411	10.0679
ITRF96	0.5796	5.2419	-3.0502	4.3120	-6.7825	35.4514	11.0945	8.0371
ITRF97	1.5487	4.2150	-1.8819	4.3840	-4.7172	33.9280	9.1012	6.4398
ITRF2000	2.8365	5.3361	-1.0826	6.5615	-4.8076	34.9179	11.3691	8.1343
Triangle 1 - 4 - 8 (Grass–Bologna - Zimmerwald) (No. 6)								
ITRF92	1.6460	8.5479	12.4127	17.1312	-3.0725	61.1010	20.2037	17.4045
ITRF93	-1.8824	7.5296	13.2853	16.3883	4.9854	67.6029	11.4029	17.1298
ITRF94	3.0908	3.2370	9.2711	10.6561	1.7058	66.8353	8.9503	10.7918
ITRF96	8.8276	-0.5745	1.6884	8.8735	1.6424	-4.5712	7.2311	9.0242
ITRF97	7.4760	2.6775	1.7626	8.5346	0.7040	21.5726	7.8306	8.5636
ITRF2000	6.1801	2.6301	0.4793	7.2081	-0.5488	21.3493	7.7569	7.2290

$$\mathbf{R} = \frac{1}{2}(\mathbf{L} - \mathbf{L}') = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial V_E}{\partial N} - \frac{\partial V_N}{\partial E}\right) \\ \frac{1}{2}\left(\frac{\partial V_N}{\partial E} - \frac{\partial V_E}{\partial N}\right) & 0 \end{bmatrix}. \quad (24)$$

With (22), (23) and (24) we could compute the geodetic strain rates of every Delaunay triangle for six ITRF realizations, and successively determine the eigenspace components (eigenvalues and eigendirections) together with the maximum shear strain rate $\varepsilon_1 - \varepsilon_2$ and the second strain rate invariant $(\varepsilon_1^2 + \varepsilon_2^2)^{1/2}$. For the purpose of the continued discussion in Section 3.5 we list only the results for two of the 11 triangles in Table 4.

3.4 The representation of the numerical results of 2-D geodetic strain rate and its interpretation

Now we can present the horizontal incremental velocities and the principal strain rates of every triangle for six ITRF realizations derived above. The interpretation of the geodetic strain also results from these six ITRF realizations and a comparison with the geodynamical setting will follow.

The incremental velocities and principal strain rates have to be represented in an appropriate way for any further interpretations and comparisons. We used the MATLAB Mapping Toolbox (MathWorks 2000) to map the surface deformation information. The *equidistant conic projection* was described by the Alexandrian astronomer, mathematician and geographer *Claudius Ptolemy* about 150 AD. Improvements were developed by *Johannes Ruysch* in 1508, *Gerardus Mercator* in the late 16th century, and *Nicolas de l'Isle* in 1745. It is also known as the *(Simple) Conic projection*. The scale is rational along all meridians and the standard parallels. It is also constant along any parallel. This projection is free of distortion along the two standard parallels. Distortion is constant along any other parallel. This projection provides a compromise in distortion between a conformal and an equal-area conic projections, of which it is neither.

The pattern of the principal strain rates (eigenvalues and eigendirections of the 2-D strain rate tensors) of 11 Delaunay-triangles and the associated incremental velocities of the selected ITRF92 to ITRF2000 sites in the study region are illustrated in Figs 4–9. *Extensionis* represented by symmetric arrows pointing outwards and *contraction* is represented by symmetric arrows pointing inwards. The incremental velocities are represented by *single* arrows.

First, let us briefly analyse the horizontal movements of these selected sites with respect to the Eurasian fixed plate. The selected six stations Grass, Onsala, Wettzell, Madrid, Graz and Zimmerwald belong to the stable European plate. They have smaller incremental velocities with respect to the Eurasian plate described by, for example, the NNR-NUVEL-1A model; in general, the incremental velocities of these six sites in ITRF97 are below the level of 1.5 mm yr⁻¹; for more details see Fig. 8. The three Italian stations, Bologna (Medicina), Matera and Noto, all show motions with respect to stable Europe. The two mainland sites, Bologna (Medicina) and Matera are east of the Apennine mountain chain and have northeast-trending velocities with rates increasing southwards from, for example, in ITRF97 from 2.9 mm yr⁻¹ to 4.8 mm yr⁻¹. In contrast, Noto on Sicily, just a few hundred km south-west of Matera is moving with rates of 4.6 mm yr⁻¹ in northwest direction at an apparent angle to the other Italian sites. These significant incremental velocities of the three sites in Italy reflect the fact that the movements of these sites, located in the plate boundary zone between Eurasia and Africa, do not agree with NNR NUVEL-1A.

Second, let us analyse the strain rate solutions of the six epochs. From the derived strain rates from (22) and the visual patterns in Figs 4 ~ 9, we can see that the magnitude and direction of the strain rates in most triangles are nearly consistent with each epoch, except for

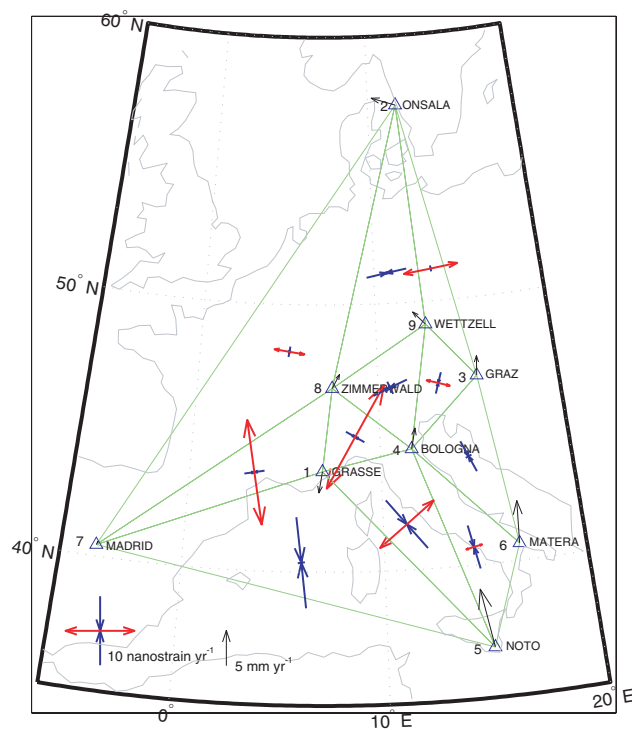


Figure 4. Pattern of the principal strain rates of 11 triangles and the associated incremental velocities of the selected ITRF92 sites in the studying region. *Extension* is represented by symmetric arrows pointing outwards and *contraction* is represented by symmetric arrows pointing inwards. The incremental velocities are represented by *single* arrows.

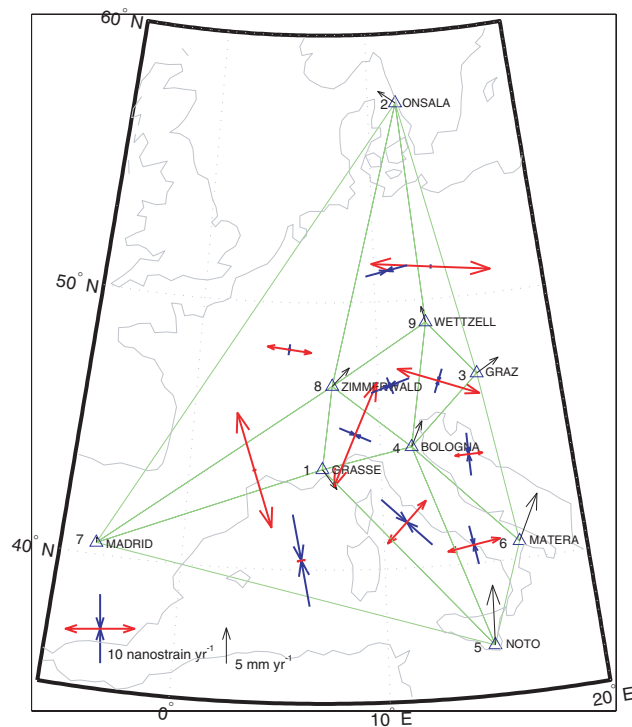


Figure 5. Pattern of the principal strain rates of 11 triangles and the associated incremental velocities of the selected ITRF93 sites in the studying region. *Extension* is represented by symmetric arrows pointing outwards and *contraction* is represented by symmetric arrows pointing inwards. The incremental velocities are represented by *single* arrows.

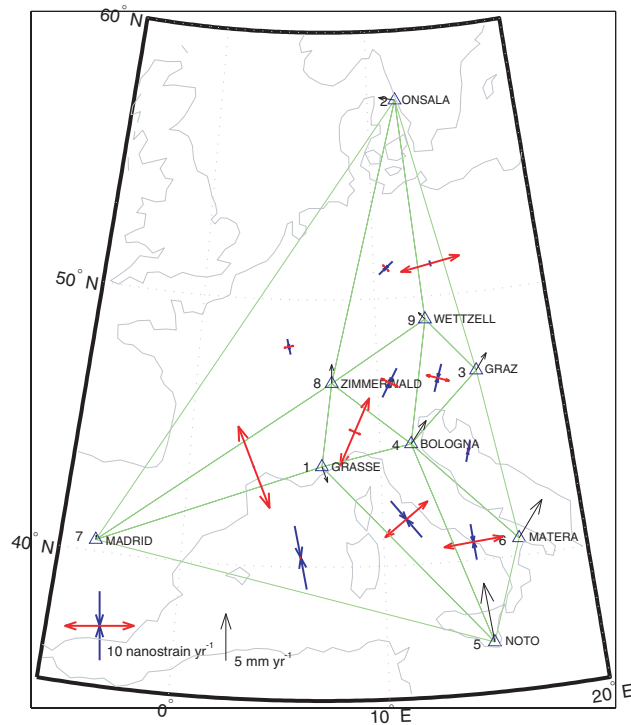


Figure 6. Pattern of the principal strain rates of 11 triangles and the associated incremental velocities of the selected ITRF94 sites in the studying region. *Extension* is represented by symmetric arrows pointing outwards and *contraction* is represented by symmetric arrows pointing inwards. The incremental velocities are represented by *single* arrows.

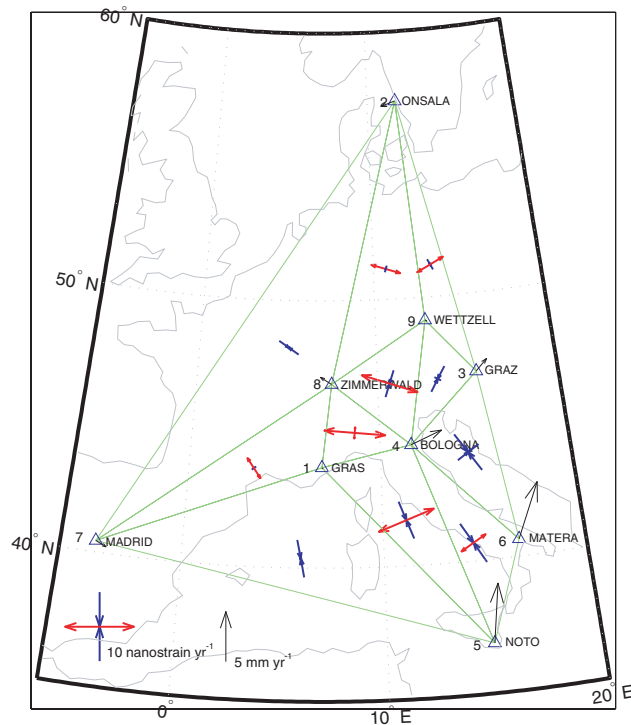


Figure 7. Pattern of the principal strain rates of 11 triangles and the associated incremental velocities of the selected ITRF96 sites in the studying region. *Extension* is represented by symmetric arrows pointing outwards and *contraction* is represented by symmetric arrows pointing inwards. The incremental velocities are represented by *single* arrows.

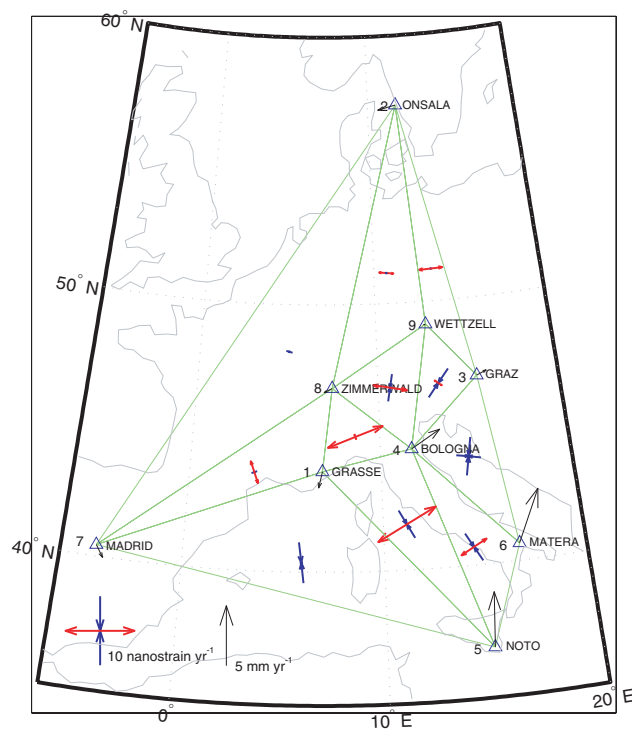


Figure 8. Pattern of the principal strain rates of 11 triangles and the associated incremental velocities of the selected ITRF97 sites in the studying region. *Extension* is represented by symmetric arrows pointing outwards and *contraction* is represented by symmetric arrows pointing inwards. The incremental velocities are represented by *single* arrows.

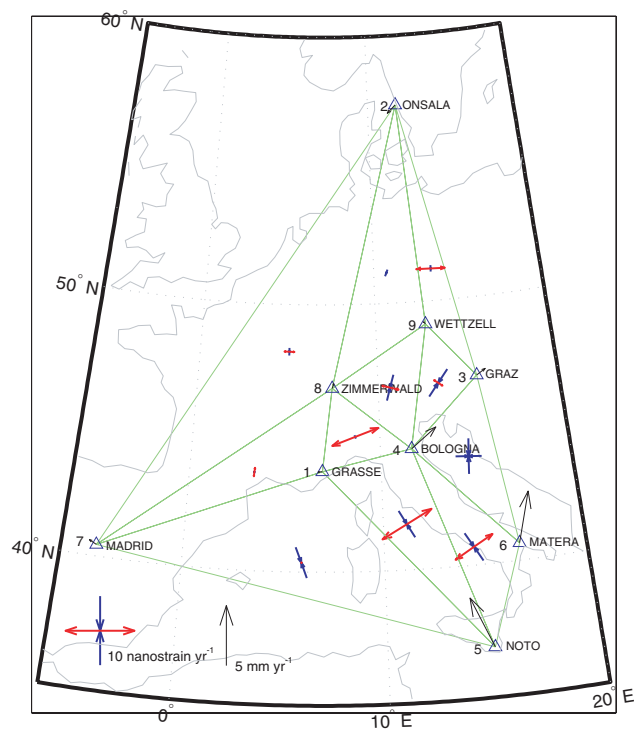


Figure 9. Pattern of the principal strain rates of 11 triangles and the associated residual velocities of the selected ITRF2000 sites in the studying region. *Extension* is represented by symmetric arrows pointing outwards and *contraction* is represented by symmetric arrows pointing inwards. The residual velocities are represented by *single* arrows.

the triangles 1-4-8 and 1-7-8. As we have explained above, the deformation pattern of this region is characterized by a complex space–time distribution of compressional and tensional events.

We have to compare these results with the geodynamic features in detail. Since this study is concentrated on the statistical inference of the eigenspace components of strain rate tensors, we limit our comparison of the intensive seismic activity in the Alpine–Mediterranean regions to the result of ITRF2000.

From Fig. 9 we can learn that the Bologna–Matera–Noto triangle (4-5-6) suffers from ENE–WSW extension strain with a rate of $5.29 \text{ nanostrain yr}^{-1}$, which consists of the seismic strain rate derived from centroid moment tensor (CMT) solutions (Pondrelli *et al.* 1995) not only in the maximum principal seismic strain rate of about $5 \text{ nanostrain yr}^{-1}$ but also in the principal direction. Furthermore, the direction of the extensional strain rate is in accordance with the tectonic evolution of this region (Apenninics), which is strongly affected by the convergence of this microplate (Ward 1994). Our geodetic strain rate results in this subregion are also in accordance with other geodynamic solutions deduced from a fault plane solution of earthquakes that occurred during the last century (Jackson & McKenzie 1988), historical seismicity (Selvaggi 1998), and the newly published geodetic results by Anzidei *et al.* (2001), Devoti *et al.* (2002a,b), Caporali (2003), Caporali *et al.* (2003) and Jiménez-Munt *et al.* (2003).

In the Western Mediterranean area, which is covered by the triangle Grass–Noto–Madrid (1-5-7), compression predominates in the NNW direction, which is in good agreement with the observed stress data (Ward 1994; Montone *et al.* 1999; Jiménez-Munt *et al.* 2003). This compression is consistent with the view that it is induced by the relative motion between Africa and Eurasia (DeMets *et al.* 1994). The geodetic E–W extension is in accordance with the extensional tectonics perpendicular to the Apenninic chain, indicated by the normal fault events. The observed extension which is perpendicular to the chain could indicate that the subduction is also active underneath the central Apennines. This pattern is in accordance with the radial stress regime proposed by Rebai *et al.* (1992).

The triangle Bologna–Matera–Graz (4-6-3) represents the strain across the Adriatic microplate. The geodetically observed North and East contractions are in good agreement with the northward motion of the Adriatic microplate with respect to Europe (Devoti *et al.* 2002a), and they are also consistent with the compressional stress pattern in this region (Müller *et al.* 1992; Montone *et al.* 1999). The northern triangle Bologna (Medicina)–Graz–Wettzell (3-4-9) represents the strain across the Eastern Alps with a smaller strain rate. The observed NNE contraction is in agreement with the main geological structures, namely the subduction of the Adriatic microplate to the Alpine front.

At the end of our comparison in the Alpine–Mediterranean regions, we present two previously published figures, namely in Fig. 10 a general geological map (Devoti *et al.* 2002b), and Fig. 11 the average of active stress map (Montone *et al.* 1999) of the Alpine–Mediterranean

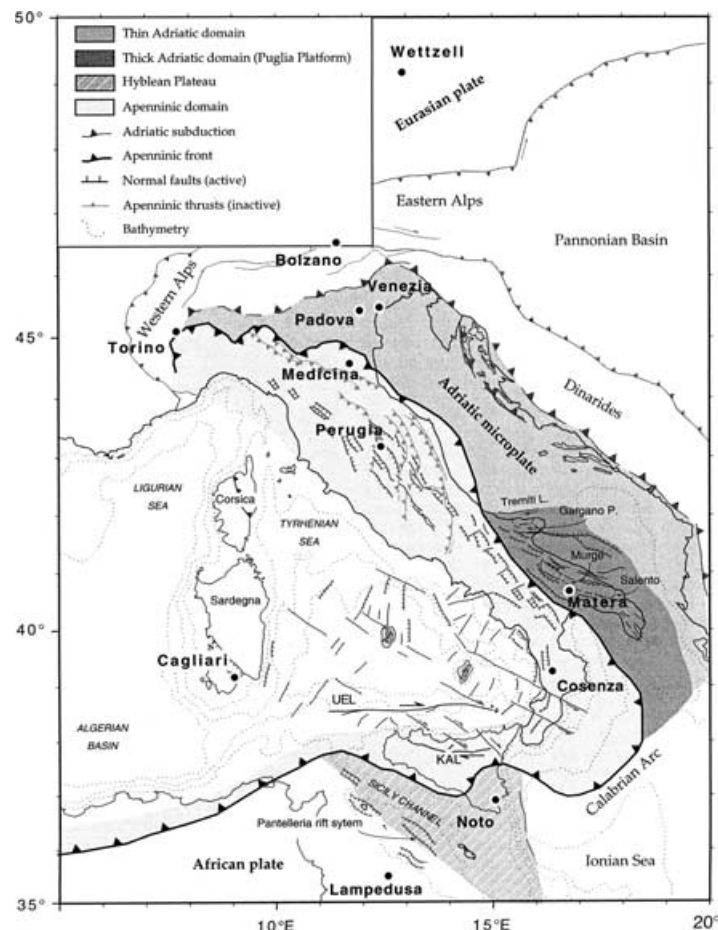


Figure 10. Simplified geological structural map of the central Mediterranean region showing the main tectonic domains (Devoti *et al.* 2002b).

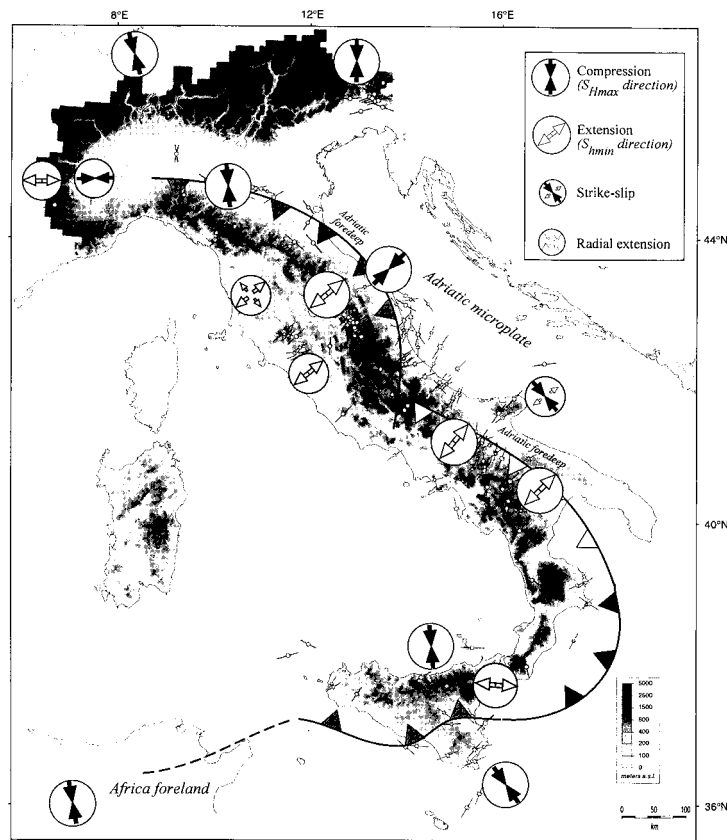


Figure 11. The average active stress map of Italy. The structural arcs with shaded triangles indicate the active compressional fronts; solid triangles show active oceanic subduction; open triangles delineate the location of Pilo-Pleistocene thrust front, presently affected by prevalent extension (Montone *et al.* 1999).

regions; both are in agreement with our geodetic strain rate results. On the other hand it has thus been shown that our results appear to be good reproductions of the geodynamical setting in these regions.

3.5 Statistical inference of the eigenspace components of 2-D strain rate tensor

In this section, as a case study, both model and hypothesis tests developed in Section 2 will be applied to the observations of random strain rate tensors, derived above for every Delaunay-triangles of the selected ITRF sites at six epochs.

3.5.1 The estimates of the eigenspace components from the strain rates observations of six epochs

With the 2-D strain rate tensor observations, calculated by (22) with the six epoch ITRF incremental velocities, we can now estimate the eigenspace components (eigenvalues and eigendirections) of the 2-D strain rate tensors, variance-covariance component matrix of type BIQUUE, and their estimated dispersion matrix with (14), (16) and (15), and successively make hypothesis tests. The detailed results of all 11 Delaunay-triangles in the study region are illustrated in Fig. 12 together with their 95 per cent confidence intervals. The estimates of the eigenspace components (eigenvalues and eigendirections) and their standard deviations together with the maximum shear strain rate $\hat{\lambda}_1 - \hat{\lambda}_2$ and the second strain rate invariant $(\hat{\lambda}_1^2 + \hat{\lambda}_2^2)^{1/2}$ for these triangles are listed in Table 5.

The 95 per cent confidence intervals for the estimated eigenvalues $\hat{\lambda}_1$, $\hat{\lambda}_2$ and eigendirections $\hat{\alpha}_1$ as illustrated in Fig. 12 provide us with a visual presentation of the possible magnitude and the directions of the extension and contraction of the strain rate. This is important for the prediction of the tectonic activity, including the possible deformation trend and its directions. For example the larger error (confidence interval) of eigenvalues and eigendirection of strain rate, detected obviously in the triangle 1-4-8 (Grasse-Bologna-Zimmerwald), results from the variety of strain rates observations of the six epochs. As illustrated in Figs 4–9, the principal direction of strain rate among the six epochs changes from NNE in ITRF92-93 and to W-E in ITRF96 and to ENE in ITRF97/ITRF2000, more detailed results are also found in Table 4. This fact reflects that the deformation pattern in this triangle is not stable during the six epochs from 1992 to 2000.

It is necessary to note that, although the strain rate tensor observations are derived from the nine ITRF sites according to the criterion discussed above, in reality they do not satisfy all the conditions of i.i.d. observations, since we have not yet found the right i.i.d. strain tensor observation sets, we apply strain rate tensor observations derived from the nine ITRF stations in six series realizations, assuming that they are at least approximately i.i.d. observations in our study.

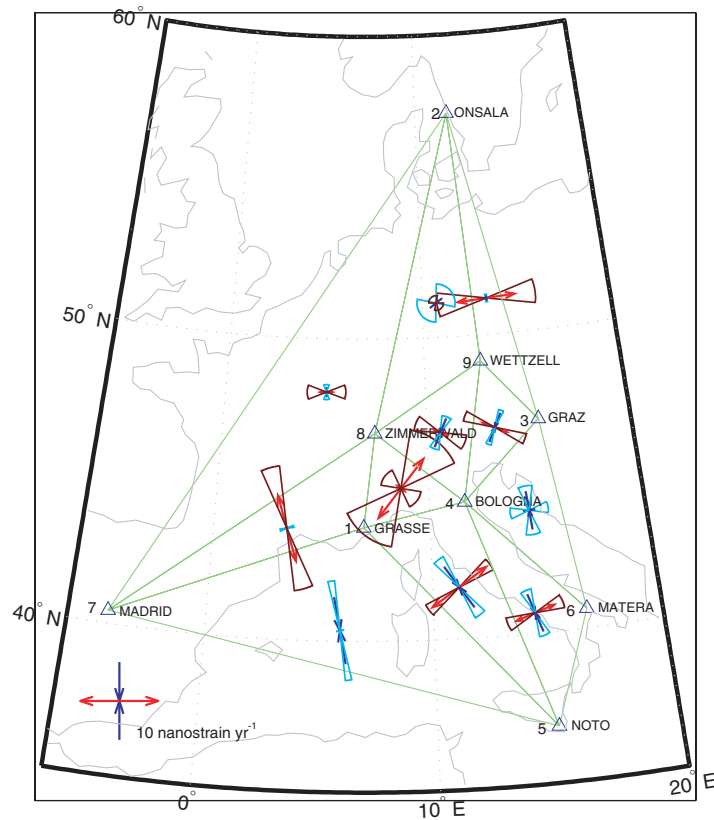


Figure 12. Eigenspace components (eigenvalues and eigendirections) of the 2-D strain rate tensors and their 95 per cent confidence intervals, estimated from the strain rate observations of the ITRF92 to ITRF2000 series in the nine triangle sites in the study region. *Extension* is represented by symmetric arrows pointing outwards and *contraction* is represented by symmetric arrows pointing inwards.

When we repeat the comparison of our strain rate pattern in Fig. 12 with the geodynamic map, see Figs 10 and 11, it can be concluded in general that our estimated of eigenspace components of a 2-D strain rate tensors are consistent with the tectonic setting. Furthermore we can benefit from the statistical information derived from the estimation procedure developed in Section 2; this will be presented in the next section.

3.5.2 Statistical inference of the estimates of eigenspace component parameters

The estimates of the eigenspace component parameters and their related dispersion matrix from the strain rate observations of six epochs reflect the statistical average information of the random strain rate tensor; utilize the advantage of the longer time span. With them we can successively perform the statistical inference.

Since we are interested in testing our statistical method, we would like to perform all the hypothesis tests discussed in Section 2 in detail for just one triangle 4-5-6 (Bologna–Noto–Matera). Using the strain rate observations in Table 4 we have estimated the eigenspace component parameters of a rank-two symmetric random tensor, their related dispersion matrix with *Theorem 2*, and the sample variance–covariance matrix $\hat{\Sigma}_y$ of type BIQUUE with *Theorem 3*; they are summarized in the Appendix A1. With these estimates of the eigenspace components of the random strain rate tensor and their dispersion matrix, and under the assumption that the observations of a symmetric rank-two random strain rate tensor are *Gauss–Laplace* normally distributed, the univariate and multivariate hypothesis tests, discussed in Section 2.3, are performed and the results are given in Appendix A2.

The above analysis show that the estimation theory about the 2-D, symmetric rank-two random tensor, developed in Section 2, respectively, by Cai *et al.* (2005), is practical when applied and produces not only estimates consistent with the tectonic setting, but also successive hypothesis tests which complete the statistical inference of eigenspace component parameters of a 2-D, symmetric rank-two strain rate tensor.

4 CONCLUSION AND OUTLOOK

Having recognized the fact that an exact distribution theory of eigenspace components of a symmetric random tensor is almost always unavailable, that is, the distributions of the eigenvalues and eigendirections of a symmetric random tensor is different from the normal distribution, and thus a direct statistical inference of them in real Engineering and Earth science problems can hardly be performed, we have investigated the statistical inference of eigenspace components of a 2-D symmetric rank-two random tensor based upon a linearized

Table 5. The estimates of the eigenspace components and their standard deviations, together with the maximum shear strain rates and the second invariant.

Triangles	Eigenspace components and standard deviations (nanostrain yr ⁻¹)						Max. shear strain rate (nanostrain yr ⁻¹)	Second invariant (nanostrain yr ⁻¹)
	$\hat{\lambda}_1$	$\hat{\sigma}_{\hat{\lambda}_1}$	$\hat{\lambda}_2$	$\hat{\sigma}_{\hat{\lambda}_2}$	$\hat{\alpha}_1$	$\hat{\sigma}_{\hat{\alpha}_1}$	$\hat{\lambda}_1 - \hat{\lambda}_2$	$\sqrt{\hat{\lambda}_1^2 + \hat{\lambda}_2^2}$
No. 1: 8–2–7 (Zim–Ons–Mad)	1.6176	±1.3012	–1.2824	±0.2501	0.1681	±8.2284	2.9000	2.0642
No. 2: 1–5–7 (Gras–Not–Mad)	–0.1637	±0.2973	–8.7116	±1.6321	9.9241	±1.4835	8.5479	8.7131
No. 3: 1–8–7 (Gras–Zim–Mad)	8.7144	±2.8800	–0.6217	±0.4370	–74.6070	±3.3523	9.3361	8.7366
No. 4: 9–8–2 (Wet–Zim–Ons)	0.2244	±0.6497	–2.0148	±1.1151	–50.7307	±19.5934	2.2392	2.0273
No. 5: 9–3–2 (Wet–Graz–Ons)	7.3502	±2.0722	–0.6027	±0.1422	8.5718	±5.4747	7.9529	7.3749
No. 6: 4–1–8 (Bol–Gras–Zim)	9.5173	±2.1984	1.1889	±1.5589	52.8733	±10.8772	8.3284	9.5913
No. 7: 4–9–8 (Bol–Wet–Zim)	1.6742	±1.9740	–3.9692	±0.1628	–21.6686	±6.4597	5.6434	4.3079
No. 8: 4–9–3 (Bol–Wet–Graz)	3.3182	±1.9163	–4.0195	±0.2443	–20.7389	±3.7226	7.3376	5.2121
No. 9: 4–3–6 (Bol–Graz–Mat)	–1.6041	±1.2931	–5.0069	±0.5130	10.1413	±6.5735	3.4028	5.2575
No. 10: 4–5–6 (Bol–Not–Mat)	5.2837	±0.9064	–5.2345	±0.3867	23.4382	±5.1932	10.5182	7.4376
No. 11: 4–1–5 (Bol–Gras–Not)	8.6068	±0.4504	–6.4977	±0.9156	37.1679	±3.8370	15.1045	10.7841

multivariate *Gauss–Markov* model, which could provide us with the second-order statistics of eigenspace components. Such a statistical inference on the estimates of eigenspace components of a random tensor is completed by the design of a linear hypothesis test.

For its linearized form of a special non-linear multivariate Gauss–Markov model for sampling the eigenspace *synthesis* of a 2-D, symmetric rank-two random tensor, the *BLUUE of the eigenspace elements* and *BIQUEE* of its variance–covariance component matrix have been established successfully by Cai *et al.* (2001) and summarized here in *Theorem 2* and *3*. The proper test statistics, such as *Hotelling's T²*, *likelihood ratio statistics* and the general linear hypothesis test with a *growth curve model*, are proposed.

As a case study both estimates BLUUE and BIQUEE and hypothesis tests have been applied successfully to the eigenspace components of 2-D strain rate tensor observations in the area of the central Mediterranean and of Western Europe, which are derived from ITRF92 to ITRF2000 series station positions and velocities in Section 3.5. The analysis with respect to geodynamical and statistical aspects shows that, in general, our estimates of the eigenspace components of a 2-D strain rate tensor are consistent with the tectonic setting in the area of the central Mediterranean and in Western Europe. Furthermore we can benefit from the statistical information derived from the estimation procedure. For example the 95 per cent confidence intervals for the estimates of eigenvalues $\hat{\lambda}_1$, $\hat{\lambda}_2$ and eigendirection $\hat{\alpha}_1$ illustrated in Fig. 12, provides us with a visual presentation of the possible magnitude and the directions of the extension and contraction of the strain rate, which is important for the prediction of the tectonic activity including the possible deformation trend and directions.

The related *linear hypothesis tests* in the case study in Section 3.5 have documented large confidence regions for the eigenspace components, namely *eigenvalues* and *eigendirections*, based upon real measurement configurations. They lead to the statement *to be cautious* with data of type extension and contraction as well as with the orientation of principal stretches.

It is necessary to note that, although the strain rate tensor observations are derived from the nine ITRF sites according to the criterion discussed above, in reality they do not satisfy all the conditions of i.i.d. observations. Since we have not yet found the right i.i.d. strain tensor observation sets, we apply strain rate tensor observations derived from the nine ITRF stations in six sequential realizations, assuming approximately that they are i.i.d. observations in our study.

In this paper we have achieved the complete solution to the statistical inference of *eigenspace components* of the deformation tensors. The models developed in the last two chapters are closed and practical, which provides us a new method in the deformation analysis. With these models we could successfully perform the statistical inference of the *eigenspace components* and the variance–covariance matrix of the *Gauss–Laplace* normally distributed observations of a random deformation tensor.

Beyond the case study in Section 3 there surely exist further applications and investigations of statistical inference for the eigenspace components of a deformation tensor.

As we have mentioned before, for the estimation of eigenspace components of type BLUUE, we need the observations of random tensors. Except for our case study with strain rate tensor observations derived from the ITRF station coordinates and their velocities in six sequential realizations, there are other types of symmetric rank-two random tensors such as stress and strain. Especially with benefit to the new development of space geodetic techniques such as GPS, the time-series observations of a network can be achieved with higher accuracy.

The time-series observations as samples enable us to apply the BLUE estimate of the eigenspace components of 2-D or 3-D random tensors and BIQUUE of their variance-covariance components matrix in a more practical way, thereby providing more realistic results.

Regarding the facts in reality, crustal motions and deformations are of 3-D nature. Most tensors in geodesy and geophysics are 3-D having been derived from geodetic, geological and seismological data. The estimation theory as developed by Cai (2004) can also be applied to the statistical analysis of the estimates of the eigenspace components of the 3-D stress/strain or seismic moment tensors with *in situ* measurements by strain meter and by seismometer, respectively, which are of focal interest in geophysics and seismology.

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APPENDIX A1

Box 1

Case study: Hypothesis test with a 2-D strain rate tensor in Triangles 4-5-6 of sites Bologna–Matera–Noto ‘the Σ -BLUE of eigenspace components of a 2-D, symmetric rank-two random tensor with *Theorem 2*’

$$\hat{\xi} = \begin{bmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\alpha} \end{bmatrix} = \begin{bmatrix} 5.2837 \text{ (n strain/y)} \\ -5.2345 \text{ (}\mu \text{ strain/y)} \\ 0.409074 \text{ (arc)} \end{bmatrix} = \begin{bmatrix} 5.2837 \text{ (n strain/y)} \\ -5.2345 \text{ (}\mu \text{ strain/y)} \\ +23^\circ 4382 \end{bmatrix}$$

‘the sample variance–covariance matrix $\hat{\Sigma}_y$ of type BIQUUE with *Theorem 3*’

$$\hat{\Sigma}_y = \begin{bmatrix} +9.908347 & -2.351484 & -2.515398 \\ -2.351484 & +1.688078 & +2.089224 \\ -2.515398 & +2.089224 & +3.449229 \end{bmatrix} \text{ (n strain / y)}^2$$

‘the related dispersion matrix of $\hat{\xi}$ with Theorem 2’

$$D(\hat{\xi}) = \Sigma_{\hat{\xi}} = \begin{bmatrix} +0.821618 & +0.208317 & -0.040791 \\ +0.208317 & +0.149544 & +0.000599 \\ -0.040791 & +0.000599 & +0.008215 \end{bmatrix}$$

APPENDIX A2

(1) Test for the eigenspace parameter vector $\xi = \xi_0$ with Σ_y unspecified with Hotelling's T^2 -test

Box 2:

Multivariate hypothesis test about the eigenspace parameter vector ξ assuming Gauss–Laplace normally distributed observations of a symmetric rank-two random strain rate tensor

First test for $\mathcal{H}_{01} : \xi = \xi_0, \mathcal{H}_{11} : \xi \neq \xi_0$ with Σ_y unspecified;

$\Leftrightarrow$

$$\mathcal{H}_{01} : \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \lambda_{10} \\ \lambda_{20} \\ \alpha_{10} \end{bmatrix} = \begin{bmatrix} 4.3840 \\ -4.7172 \\ 0.5922 \end{bmatrix}, \mathcal{H}_{11} : \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \alpha_1 \end{bmatrix} \neq \begin{bmatrix} \lambda_{10} \\ \lambda_{20} \\ \alpha_{10} \end{bmatrix} = \begin{bmatrix} 4.3840 \\ -4.7172 \\ 0.5922 \end{bmatrix} \text{ with } \Sigma_y \text{ unspecified}$$

‘Hotelling's T^2 statistic’ (Hotelling 1931; Muirhead 1982; Rencher 1998)

$$T^2 := [\hat{\xi} - \xi_0]' \hat{\Sigma}_{\hat{\xi}}^{-1} [\hat{\xi} - \xi_0]$$

with respect to the eigenspace components of type Σ -BLUE and the dispersion matrix $\hat{\Sigma}_{\hat{\xi}}$

$$\hat{\xi} = \begin{bmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\alpha}_1 \end{bmatrix} = \begin{bmatrix} 5.2837 \text{ (n strain/y)} \\ -5.2345 \text{ (n strain/y)} \\ 0.409074 \text{ (arc)} \end{bmatrix}, \hat{\Sigma}_{\hat{\xi}} = \begin{bmatrix} +0.821618 & +0.208317 & -0.040791 \\ +0.208317 & +0.149544 & +0.000599 \\ -0.040791 & +0.000599 & +0.008215 \end{bmatrix}.$$

According to the design of the test for the eigenspace parameter vector in Section 2 at the error probability α we reject $\mathcal{H}_{01}$ in favor of $\mathcal{H}_{11}$ if

$$T^2 = [\hat{\xi} - \xi_0]' \hat{\Sigma}_{\hat{\xi}}^{-1} [\hat{\xi} - \xi_0] > \frac{(n-1) \cdot 3}{n-3} F_{3, n-3}(1-\alpha) = T_{1-\alpha}^2.$$

With error probability $\alpha = 5$ per cent and $n = 6$ the critical value

$$F_{3, n-3}(1-\alpha) = 9.28, \quad T_{1-\alpha}^2 = 46.38.$$

Since Hotelling's $T^2 = 7.32 < T_{1-\alpha}^2 = 46.38$, accordingly we accept the null hypothesis $\mathcal{H}_{01} : \xi = \xi_0$ with the risk of $\alpha = 5$ per cent of a Type I error.

(2) Test for a distinct element of the eigenspace parameter vector with Student t -test

Box 3

Separate Student t -tests about the eigenspace parameters in ξ

$$\begin{aligned} \text{Second test for (separately)} \quad & \mathcal{H}_{02} : \lambda_1 = \lambda_{10} = 4.3840 \mid \lambda_2 = \lambda_{20} = -4.7172 \mid \alpha_1 = \alpha_{10} = 0.5922 \\ & \mathcal{H}_{12} : \lambda_1 \neq \lambda_{10} = 4.3840 \mid \lambda_2 \neq \lambda_{20} = -4.7172 \mid \alpha_1 \neq \alpha_{10} = 0.5922 \end{aligned}$$

‘two-sided tests with the test quantities’

$$t_1 := \frac{\hat{\lambda}_1 - \lambda_{10}}{\hat{\sigma}_1}, \quad t_2 := \frac{\hat{\lambda}_2 - \lambda_{20}}{\hat{\sigma}_2}, \quad t_3 := \frac{\hat{\alpha}_1 - \alpha_{10}}{\hat{\sigma}_3}$$

with respect to $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\alpha}_1$ of type Σ -BLUE and their variances from Box 1. t_1, t_2 and t_3 are elements of the Student t -distribution with $n-1$ degree of freedom.

With error probability $\alpha = 5$ per cent we derive

$$t_{1, 1-\alpha/2} = t_{2, 1-\alpha/2} = t_{3, 1-\alpha/2} = +2.57$$

$$t_{1, \alpha/2} = t_{2, \alpha/2} = t_{3, \alpha/2} = -2.57.$$

The critical values

$$c_{\lambda_1, \alpha/2} = \hat{\sigma}_{\lambda_1} t_{1, \alpha/2} + \lambda_{10} = 2.0539, \quad c_{\lambda_1, 1-\alpha/2} = \hat{\sigma}_{\lambda_1} t_{1, 1-\alpha/2} + \lambda_{10} = 6.7141$$

$$c_{\lambda_2, \alpha/2} = \hat{\sigma}_{\lambda_2} t_{1, \alpha/2} + \lambda_{20} = -5.7113, \quad c_{\lambda_2, 1-\alpha/2} = \hat{\sigma}_{\lambda_2} t_{2, 1-\alpha/2} + \lambda_{20} = -3.7231$$

$$c_{\alpha_1, \alpha/2} = \hat{\sigma}_{\alpha_1} t_{1, \alpha/2} + \alpha_{10} = 0.3592, \quad c_{\alpha_1, 1-\alpha/2} = \hat{\sigma}_{\alpha_1} t_{3, 1-\alpha/2} + \alpha_{10} = 0.8251$$

indicate the confidence intervals

$$c_{\lambda_1, \alpha/2} = 2.0539 < \hat{\lambda}_1 = 5.2837 < c_{\lambda_1, 1-\alpha/2} = 6.7141,$$

$$c_{\lambda_2, \alpha/2} = -5.7113 < \hat{\lambda}_2 = -5.2345 < c_{\lambda_2, 1-\alpha/2} = -3.7231,$$

$$c_{\alpha_1, \alpha/2} = 0.3592 < \hat{\alpha}_1 = 0.409074 < c_{\alpha_1, 1-\alpha/2} = 0.8251,$$

$$(c_{\alpha_1, \alpha/2} = 20^\circ 34' 50''.32 < \hat{\alpha}_1 = 23^\circ 26' 17''.52 < c_{\alpha_1, 1-\alpha/2} = 47^\circ 16' 29''.09),$$

thereby suggesting the acceptance of all three null hypotheses.

$$\mathcal{H}_{02}: \lambda_1 = \lambda_{10} = 4.3840, \lambda_2 = \lambda_{20} = -4.7172, \alpha_1 = \alpha_{10} = 0.5922$$

with the risk of $\alpha = 5$ per cent of a *Type I error*.

The 95 per cent *confidence interval* for the eigenvalues λ_1 , λ_2 and the eigendirection α_1 are

$$[2.0539, 6.7141](\mu \text{ strain/y});$$

$$[-5.7113, -3.7231](\mu \text{ strain/y});$$

$$[20^\circ 34' 50''.32, 47^\circ 16' 29''.09]$$

respectively.

(3) Test for the variance–covariance matrix $\Sigma_y = \Sigma_0$ with the *likelihood ratio test*

Box 4

Multivariate hypothesis tests about the variance–covariance matrix Σ_y

Fourth test for $\mathcal{H}_{03}: \Sigma_y = \Sigma_0, \mathcal{H}_{13}: \Sigma_y \neq \Sigma_0$

‘unbiased modified likelihood ratio statistic Λ_1 ’ (Giri 1977; Muirhead 1982; Koch 1999)

$$\Lambda_1 = \left(\frac{e}{n-1}\right)^{3(n-1)/2} (\det(n-1)\hat{\Sigma}_y \Sigma_0^{-1})^{(n-1)/2} \text{etr}\left\{-\frac{1}{2}(n-1)\hat{\Sigma}_y \Sigma_0^{-1}\right\}$$

with respect to the sample variance–covariance matrix $\hat{\Sigma}_y$ of type BIQUUE in *Table 2*. Since our sample size is relatively small we have to use the exact distribution of $-2 \log \Lambda_1$, whose upper 5 and 1 percentage points have been provided by Muirhead (1982, p. 360).

With $\alpha = 5$ per cent the *critical value* is

$$L_{1-\alpha} = -2 \log \Lambda_1(1 - \alpha) = 15.508$$

Let us perform this test with three different variance–covariance matrices Σ_{01} , Σ_{02} and Σ_{03} , namely

$$\Sigma_{01} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Sigma_{02} = \begin{bmatrix} 2.033987 & 0.444505 & 0.033115 \\ 0.444505 & 0.982429 & 0.585175 \\ 0.033115 & 0.585175 & 1.02023 \end{bmatrix},$$

$$\Sigma_{03} = \begin{bmatrix} 9.808347 & -2.351484 & -2.515398 \\ -2.351484 & 1.588078 & 2.089224 \\ -2.515398 & 2.089224 & 3.349229 \end{bmatrix},$$

which are chosen according to the hypotheses that the *variance–covariance component matrix* Σ_y is equal to (1) a unit matrix Σ_{01} ; (2) Σ_{02} – the variance–covariance matrix of the strain rate observations derived from ITRF97 incremental velocities; and (3) a matrix Σ_{03} whose diagonal elements are 0.1 smaller than the estimated variance–covariance component matrix $\hat{\Sigma}_y$ as given in *Box 1*.

With respect to the likelihood ratio statistics Λ_1 and the related $-2 \log \Lambda_1 =: L_1$ we find:

$$\Lambda_{11} = 2.264853 \times 10^{-11} \quad L_{11} = -2 \log \Lambda_{11} = 49.021852$$

$$\Lambda_{12} = 1.117952 \times 10^{-6} \quad L_{12} = -2 \log \Lambda_{12} = 27.408023$$

$$\Lambda_{13} = 0.694227 \quad L_{13} = -2 \log \Lambda_{13} = 0.729913$$

Since $L_{11} = 49.021852 > L_{1-\alpha} = 15.805$, we reject the *null hypothesis* $\mathcal{H}_{03}: \Sigma_y = \Sigma_{01}$ with the risk of $\alpha = 5$ per cent of a *Type I error*.

Since $L_{12} = 27.408023 > L_{1-\alpha} = 15.805$, we reject the *null hypothesis* $\mathcal{H}_{03}: \Sigma_y = \Sigma_{02}$ with the risk of $\alpha = 5$ per cent of a *Type I error*.

Since $L_{13} = 0.729913 < L_{1-\alpha} = 15.805$, we accept the *null hypothesis* $\mathcal{H}_{03} : \Sigma_y = \Sigma_{03}$ with the risk of $\alpha = 5$ per cent of a *Type I error*.

(4) Test for the eigenspace parameter vector and variance–covariance matrix $\xi = \xi_0, \Sigma_y = \Sigma_0$ *likelihood ratio test*

Box 5

Multivariate hypothesis tests about the eigenspace parameter vector ξ and the variance–covariance matrix Σ_y

Fifth test for $\mathcal{H}_{04} : \xi = \xi_0, \Sigma_y = \Sigma_0, \mathcal{H}_{14} : \xi \neq \xi_0$ or $\Sigma_y \neq \Sigma_0$

‘unbiased likelihood ratio statistic Λ_2 ’ (Anderson 1984; Muirhead 1982)

$$\Lambda_2 = \left(\frac{e}{n}\right)^{3n/2} (\det(n-1)\hat{\Sigma}_y \Sigma_0^{-1})^{n/2} \text{etr}\left\{-\frac{1}{2}(n-1)\hat{\Sigma}_y \Sigma_0^{-1}\right\} \exp\left\{-\frac{1}{2}[\hat{\xi} - \xi_0]' \Sigma_{\xi_0}^{-1} [\hat{\xi} - \xi_0]\right\}$$

with respect to the eigenspace components of type Σ -BLUUE and variance–covariance matrix $\hat{\Sigma}_y$ of type BIQUUE in *Box 1* and $\Sigma_{\xi_0}^{-1} = (1/n)(A' \Sigma_0^{-1} A)^{-1}$. Since our sample size is relatively small we have to use the exact distribution of $-2 \log \Lambda_2$, whose upper 5 and 1 percentage points have been provided by Muirhead (1982, p. 371).

With $\alpha = 5$ per cent the *critical value* is derived

$$L_{1-\alpha} = -2 \log \Lambda_2(1 - \alpha) = 24.431.$$

Choose ξ_0 as in the hypothesis test one and Σ_0 as Σ_{03} in the fourth hypothesis test, namely

$$\xi_0 = \begin{bmatrix} \lambda_{10} \\ \lambda_{20} \\ \alpha_{10} \end{bmatrix} = \begin{bmatrix} 4.3840 \\ -4.7172 \\ 0.5922 \end{bmatrix}, \quad \Sigma_0 = \begin{bmatrix} 9.808347 & -2.351484 & -2.515398 \\ -2.351484 & 1.588078 & 2.089224 \\ -2.515398 & 2.089224 & 3.349229 \end{bmatrix}.$$

With respect to the likelihood ratio statistics Λ_2 and related $-2 \log \Lambda_2 = L_2$ data we are led to

$$\Lambda_2 = 0.008752 L_2 = -2 \log \Lambda_2 = 9.476949$$

Since $L_2 = 9.476949 < L_{1-\alpha} = 24.431$, we accept the *null hypothesis* $\mathcal{H}_{04} : \xi = \xi_0, \Sigma_y = \Sigma_0$ with the risk of $\alpha = 5$ per cent of a *Type I error*.

The next two hypothesis tests will not be discussed here in detail. We refer for (5) *Eigen inference* about the orthonormally transformed parameters η with *Student t-test* to Cai *et al.* (2005) and for the (6) general linear hypothesis test with a *growth curve model* for eigenspace parameters to Cai (2004).

APPENDIX A3 Some Notations

$\mathbb{T}_0^2$	(2, 0) Tensor space
$\mathcal{T}$	A symmetric 2-D rank-two random strain tensor
$\{\mathbf{e}^1, \mathbf{e}^2\}$	orthonormal base vector
$\mathbf{e}^i \otimes \mathbf{e}^j$	The tensor product of base vector $\mathbf{e}^i$ and $\mathbf{e}^j$
$\mathbf{T} = [t_{ij}]$	A matrix representation of the 2-D symmetric rank-two tensor
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^{2 \times 2}$	The set of 2×2 real matrices
$\mathbf{U}$	The orthonormal matrix
$]a, b[$	Half-open interval (open on the left)
$\text{vec } \mathbf{A}$	The vec of a matrix $\mathbf{A}$, which stacks the columns one on top of the other to form a large column vector
$\text{vech } \mathbf{A}$	The vech of a matrix $\mathbf{A}$ (read vec half), which is a generalization of vec operator for symmetric matrices
E	The expectation operation
$D\{\mathbf{Y}\}$	The dispersion function of $\mathbf{Y}$
Σ_y	The variance–covariance matrix of the vector $\mathbf{y}$
$\mathbf{I}_n \otimes \Sigma_y$	<i>Kronecker–Zehfuss</i> product of matrices $\mathbf{I}_n$ and Σ_y
$\mathcal{O}$	The big $\mathcal{O}$ notation to represent the order