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Spreading of a NAPL lens in a double-porosity medium

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Abstract Spreading of a non-aqueous phase liquid (NAPL) denser than water (DNAPL) lens (mound) in the unsaturated zone of double-porosity aquifer above an impervious plane boundary is investigated. The double-porosity aquifer is conceptualized as a fracture network surrounding pervious blocks. Vertical gravity equilibrium is assumed to prevail in each one of the two media, fractures and blocks. Through vertical integration, two coupled partial differential equations for the DNAPL content in each medium are obtained. The mass exchange rate between high- and low-permeability media is considered as a function of NAPL content. The dominant effect is gravity, whereas capillary forces are negligible. Analytical solutions for one-dimensional and axisymmetric problems of mound spreading are obtained.

Keywords aquifer contamination · double-porosity medium · NAPL mound spreading · unsaturated zone

Abbreviations

DNAPL	NAPL denser than water
HP	high-permeability medium
LP	low-permeability medium
NAPL	non-aqueous phase liquid
VE	vertical gravity-capillary equilibrium

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Nomenclature

ϵ	relative volume of fractures above the unit of horizontal surface
$\phi_i,\ i = 1, 2,$	porosity in fractures and matrix, respectively
h_i	height of the NAPL in both zones
$\mathcal{U}$	$\phi_1 h_1 S_{a1}$
$\mathcal{V}$	$\phi_2 h_2 S_{a2}$
$q(x, y, t)$	sink term in the equation of mass conservation
C_{H} and C_{L}	constants in the equation of mass conservation
Ω	amount of NAPL above the unit of horizontal surface
k_1 and k_2	the permeabilities of HP and LP domains
μ_{n} and μ_{w}	NAPL and water viscosity
g	acceleration due to gravity
$\Delta\rho = \rho_{\mathrm{n}} - \rho_{\mathrm{w}}$	density difference between DNAPL and water
y	horizontal coordinate axis
x	coordinate axis aligned with the direction of maximum slope
p^*	characteristic capillary pressure between phases
Λ	$\frac{g\Delta\rho}{\delta\mu}$
u	$\mathcal{U}^2$
v	$\mathcal{V}^2$
p	Laplace transform parameter
$U(p)$	Laplace transform of $u(x, t)$
$V(p)$	Laplace transform of $v(x, t)$
r	radial coordinate
$A(p)$	Laplace transform of $u(0, t)$

1 Introduction

Spreading of a non-aqueous phase liquid (NAPL) contaminant lens in a homogeneous porous medium was presented in some earlier publications [2, 5, 6]. This paper deals with the spreading of NAPL denser than water (DNAPL) in a double-porosity aquifer. The porous-fractured rocks, consisting of a system of porous blocks, separated by a system of thin fissures, are often encountered in water-bearing formations. Fluid flow in porous-fractured soils presents a complicated problem for hydrodynamic modeling being accompanied by active non-equilibrium phenomena. Two types of models are mostly used for the description of flow in porous-fractured media: model with a regular system of fractures and double-porosity model. The double-porosity model of porous-fractured rock, accepted in this paper, is conceptualized as two overlapping continuous media coupled through transfer of fluids between them [1]. The same approach and models can be used for the description of a water mound with free surface spreading in fractured-porous medium over a plane impermeable surface.

2 Spreading of a NAPL mound in a double-porosity aquifer

We consider the movement of a DNAPL mound (lens) in a water-saturated double-porosity medium over a plane horizontal or sloping impermeable surface.

Let the average by height of the relative volume of high-permeability (HP) medium (fractures) over a unit of impermeable boundary be ϵ, and the corresponding value for low-permeability (LP) medium (matrix) be $1-\epsilon$. Then the amount of NAPL above the unit of horizontal surface, Ω, is

$$\Omega(x, y, t) = \epsilon\mathcal{U}(x, y, t) + (1-\epsilon)\mathcal{V}(x, y, t), \tag{1}$$

where

$$\mathcal{U} = \phi_1 h_1 S_{a1}, \quad \mathcal{V} = \phi_2 h_2 S_{a2}; \tag{2}$$

ϕ_i $(i = 1, 2)$ are the porosities of HP and LP media, respectively, $h_i(x, y, t)$ are the heights of the zones in each medium where the DNAPL is present and S_{ai} is the NAPL saturation averaged by h_i.

In the pores of each continuous medium (fractures and matrix), the vertical gravity-capillary equilibrium (VE) between two phases (air and water or water and DNAPL) is assumed.

Then in the DNAPL transport equations for HP medium or fractures system integrated by vertical as derived in [6], we must add a sink term $q(x, y, t)$ equal to the rate of mass exchange between HP and LP domains, and add an equal source term to the transport equation in LP medium. The value of q is assumed to be positive if the fluid flow is directed from HP to LP domain. The resulting equations are

$$\epsilon\frac{\partial \mathcal{U}}{\partial t} - C_{\mathrm{H}}\frac{\partial}{\partial x}T_{\mathrm{H}}(\mathcal{U})\left[\left(\frac{\partial \Phi_{\mathrm{H}}(\mathcal{U})}{\partial x} - i_{\mathrm{o}}\right)\right] - C_{\mathrm{H}}\frac{\partial}{\partial y}\left(T_{\mathrm{H}}(\mathcal{U})\frac{\partial \Phi_{\mathrm{H}}(\mathcal{U})}{\partial y}\right) = -q \tag{3}$$

for the HP medium and

$$(1-\epsilon)\frac{\partial \mathcal{V}}{\partial t} - C_{\mathrm{L}}\frac{\partial}{\partial x}T_{\mathrm{L}}(\mathcal{V})\left[\left(\frac{\partial \Phi_{\mathrm{L}}(\mathcal{V})}{\partial x} - i_{\mathrm{o}}\right)\right] - C_{\mathrm{L}}\frac{\partial}{\partial y}\left(T_{\mathrm{L}}(\mathcal{V})\frac{\partial \Phi_{\mathrm{L}}(\mathcal{V})}{\partial y}\right) = q \tag{4}$$

for the LP medium. The constants C_{H} and C_{L} are equal to

$$C_{\mathrm{H}} = \frac{k_1\rho_{\mathrm{n}}g}{\phi\mu_{\mathrm{n}}}, \quad C_{\mathrm{L}} = \frac{k_2\rho_{\mathrm{n}}g}{\phi\mu_{\mathrm{n}}} \tag{5}$$

where k_1 and k_2 are the permeabilities of HP and LP domains, respectively, μ_{n} and μ_{w} are the NAPL and water viscosity, respectively, g is the acceleration due to gravity and $\Delta\rho = \rho_n - \rho_w$ is the density difference between DNAPL and water. The y-axis is assumed to be horizontal, and the x-axis is assumed to be aligned with the direction of maximum slope i_{o} (which is presumed to be small).

We consider the situation when the dominating driving force for the mound spreading is the gravity rather than capillarity, implying that $\mathcal{U}$ and $\mathcal{V}$ are much greater than $p^*/\Delta\rho g$ (where p^* is a characteristic capillary pressure between phases). Then T_{L}, T_{H} , Φ_{L} and Φ_H are linear functions of arguments $\mathcal{U}$ and $\mathcal{V}$, respectively.

If $k_2 \ll k_1$, the flow in LP domain may be neglected, and Eq. 4 reduces to

$$(1-\epsilon)\frac{\partial \mathcal{V}}{\partial t} = q. \tag{6}$$

The mass exchange rate q, averaged by z, is considered a function of the values of $\mathcal{U}$ and $\mathcal{V}$. Neglecting capillarity, we accept that the mass exchange from fractures to matrix is due to gravity and can be approximated as proportional to the difference of fluid levels in these domains.

Hereafter we put $T_{\mathrm{H}}(\mathcal{U}) = T_{\mathrm{o}}\mathcal{U}$ and $\Phi_{\mathrm{H}}(\mathcal{U}) = \Phi_{\mathrm{o}}\mathcal{U}$. Equations 3 and 6 become:

$$q = \Lambda(\mathcal{U} - \mathcal{V}), \quad \Lambda \equiv \frac{g\Delta\rho}{\delta\mu} \tag{7}$$

and

$$\epsilon\frac{\partial\mathcal{U}}{\partial t} - C^*\left(\frac{\partial^2\mathcal{U}^2}{\partial x^2} + \frac{\partial^2\mathcal{U}^2}{\partial x^2}\right) + 2C^*\mathcal{U}i_o + \Lambda(\mathcal{U}-\mathcal{V}) = 0 \tag{8}$$

3 Linearized model

We will use the system of Eqs. 7 and 8 in approximate linearized form. The linearization may be performed by different ways. One is to assume that the differences between the functions $\mathcal{U}$ and $\mathcal{V}$ and their mean levels $\mathcal{U}_{\mathrm{o}}$ and $\mathcal{V}_{\mathrm{o}}$ are small. However, the most appropriate linearization principle (see [7]) is to change the unknown functions from $\mathcal{U}$ and $\mathcal{V}$ to $u = \mathcal{U}^2$ and $v = \mathcal{V}^2$ and to approximate the functions $\sqrt{u}$ and $\sqrt{v}$ by linear ones. The resulting linearized equations remain the same except for the constant parameters. Let u and v be the variables in linearized equations.

We assume the exchange rate q proportional to $u - v$:

$$q = \beta(u - v),$$

then Eqs. 6 and 8 yield

$$\epsilon\frac{\partial u}{\partial t} - D_o\frac{\partial^2 u}{\partial x^2} + D_1 i_o\frac{\partial u}{\partial x} + \beta(u - v) = 0, \tag{9}$$

and

$$(1 - \epsilon)\frac{\partial v}{\partial t} - \beta(u - v) = 0, \tag{10}$$

where $D_{\mathrm{o}} = 0.5C^* T_{\mathrm{o}}\Phi_{\mathrm{o}}$, D_1 is proportional to C^* and β is proportional to Λ with constants depending on the linearization method.

Two limiting cases of the mass exchange rate β are to be noted. If $\beta = 0$, v is a constant equal to its initial value ($v = 0$); in other words, there is no inflow into matrix, and Eq. 9 becomes a linear diffusion (heat-transfer) equation with an equivalent diffusivity coefficient D_* equal to D_{o}/ϵ. In the opposite limiting case $\beta \to \infty$ (instantaneous mass exchange), from Eq. 10, it follows that $u = v$, and the sum of Eqs. 9 and 10 also gives for u a linear diffusion equation with equivalent coefficient equal to D_{o}.

3.1 Spreading of a lens over horizontal surface

If the impermeable surface is horizontal, $i_o = 0$. Equation 9 becomes

$$\epsilon\frac{\partial u}{\partial t} - D_{\mathrm{o}}\frac{\partial^2 u}{\partial x^2} + \beta(u - v) = 0. \tag{11}$$

We consider the following two cases of a mound spreading over a horizontal surface: (1) a mound generated from a point source with a constant value of u at $x = 0$; and (2) a mound generated from a short-time acting source, when a finite quantity of NAPL is injected in a point of (HP) medium.

Applying the Laplace transform with parameter p to Eqs. 10 and 11, we obtain after some manipulations the equations for functions $U(x, p)$ and $V(x, p)$, the Laplace transformations of $u(x, t)$ and $v(x, t)$:

$$D_*\frac{\mathrm{d}^2 U}{\mathrm{d}x^2} = \left(\epsilon p + \frac{(1-\epsilon)\beta p}{(1-\epsilon)p + \beta}\right)U \tag{12}$$

and

$$V = \frac{\beta U}{(1-\epsilon)p + \beta} \tag{13}$$

The solution of Eq. 12 subject to condition $U \to 0$ at $t \to \infty$ is

$$U = A(p)\exp\left[-\sqrt{p\left(\epsilon + \frac{\beta(1-\epsilon)}{(1-\epsilon)p + \beta}\right)}\frac{x}{\sqrt{D_*}}\right] \tag{14}$$

where $A(p)$ is the Laplace transform of $u(0, t)$. For problem (1), when $u(0, t) = u_{\mathrm{o}}$ =const, $A_1(p) = u_{\mathrm{o}}/p$.

The function $U = U_1(p)$ is

$$U_1(p) = \frac{u_{\mathrm{o}}}{p}\exp\left[-\sqrt{p\left(\epsilon + \frac{\beta(1-\epsilon)}{(1-\epsilon)p + \beta}\right)}\frac{x}{\sqrt{D_*}}\right]. \tag{15}$$

The inverse of Laplace transform gives for $u(x, t)$:

$$u(x, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{e}^{pt} U_1(p)\mathrm{d}p, \tag{16}$$

where p is a complex variable, $p = -\sigma + i\tau$.

For the case $\epsilon = 0$, the equations, identical to Eqs. 9 and 10, describe also the fluid flow into a gallery in slightly compressible fractured-porous medium [1, p. 188]. Using the same considerations as G.I. Barenblatt in the cited book, the integral in Eq. 16 can be transformed into the integral in the p - plane, enveloping the negative part of the real axis, $0 < \sigma < \infty$, and the integral encircling the origin, (0, 0).

The second integral gives a constant. The first one differs from zero only if the expression under the radical sign in Eq. 15 is negative, i.e. at

$$0 < \sigma < \frac{\beta}{1-\epsilon} \quad \text{and} \quad \sigma > \frac{\beta}{\epsilon(1-\epsilon)}. \tag{17}$$

After some manipulations and change of variables, the function $u(x,t)$ may be presented as

$$u(x,t) = 1 - \frac{1}{\pi}S_1 - \frac{1}{\pi}S_2 \tag{18}$$

where

$$S_1 = 2\beta z^2 \int_0^\infty \exp\left(-\frac{\beta t}{(1-\epsilon)}\frac{\zeta^2}{(\beta z^2+\zeta^2)}\right) \times \sin\left(\zeta\sqrt{\frac{\epsilon}{(1-\epsilon)}\frac{\beta z^2}{(\beta z^2+\zeta^2)}+1}\right)\frac{\mathrm{d}\varsigma}{\varsigma(\beta z^2+\varsigma^2)} \tag{19}$$

and

$$S_2=\int_{\beta/\epsilon(1-\epsilon)}^{\infty} \exp(-\sigma t)\sin\left(\sqrt{\epsilon\sigma - \frac{\beta(1-\epsilon)\sigma}{\sigma(1-\epsilon)-\beta}}\,\frac{x}{D_*}\right)\frac{\mathrm{d}\sigma}{\sigma} \tag{20}$$

where $z = x/D_*$.

To calculate $v(x,t)$, we can use expressions (13) and (16) and obtain

$$v(x,t) = 1 - \frac{1}{\pi}T_1 - \frac{1}{\pi}T_2 \tag{21}$$

where

$$T_1 = \int_0^{\beta/(1-\epsilon)} \exp(-\sigma t)\sin\left(\sqrt{\epsilon\sigma + \frac{\beta(1-\epsilon)\sigma}{\beta-\sigma(1-\epsilon)}}\,\frac{x}{D_*}\right) \times \frac{\mathrm{d}\sigma}{\sigma(\beta-\sigma(1-\epsilon))} \tag{22}$$

and

$$T_1 = 2\int_0^\infty \exp\left(-\frac{\beta t}{(1-\epsilon)}\frac{\zeta^2}{(\beta z^2+\zeta^2)}\right) \times \sin\left(\zeta\sqrt{\frac{\epsilon}{(1-\epsilon)}\frac{\beta z^2}{(\beta z^2+\zeta^2)}+1}\right)\frac{\mathrm{d}\varsigma}{\varsigma}. \tag{23}$$

The curves $u(\xi)$ and $v(\xi)$, where $\xi = x/D_*$ calculated using the above equations for five successive values of dimensionless time t, are shown in Fig. 1 for the values of parameters $\beta = 1$ and $\epsilon = 0.2$.

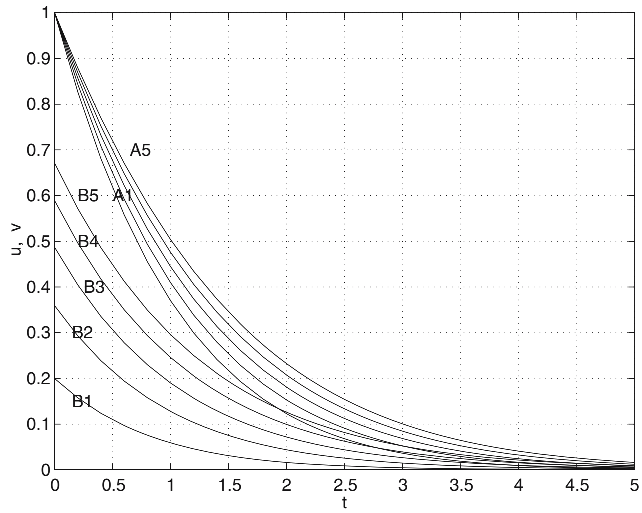

Fig. 1 A1–A5 curves $u(x,t)$ for the values of t from 0.2 to 1. B1–B5 curves $v(x,t)$ for the same values of t

Next we discuss the solution of Eqs. 9 and 10 for the case of a source acting for a limited time interval. We start from the fundamental solution of the diffusion equation, corresponding to an instantaneous source.

This solution can be written as

$$u(x,t) = \frac{\exp\left(-\frac{x^2}{4D_* t}\right)}{\sqrt{\pi t}}. \tag{24}$$

Equation 24 corresponds to the boundary condition $u(0,t) = 1/\sqrt{\pi t}$ at $x = 0$ where D_* is the diffusivity coefficient. Using the same boundary condition for Eq. 10, or, in Laplace variables, $U(0,p) = 1/\sqrt{p}$ expression (15) becomes

$$U(p) = \frac{B}{\sqrt{p}}\exp\left[-\sqrt{p\left(\epsilon + \frac{\beta(1-\epsilon)}{(1-\epsilon)p+\beta}\right)}\frac{x}{\sqrt{D_*}}\right]. \tag{25}$$

The inverse yields:

$$u(x,t) = \frac{1}{\pi}Q_1 + \frac{1}{\pi}Q_2 \tag{26}$$

where

$$Q_1(x,t) = \frac{1}{2}\int_0^{\beta/(1-\epsilon)} \exp(-\sigma t) \times \cos\left(z\sqrt{\epsilon\sigma + \frac{\beta(1-\epsilon)\sigma}{\beta-(1-\epsilon)\sigma}}\right)\frac{\mathrm{d}\sigma}{\sqrt{\sigma}} \tag{27}$$

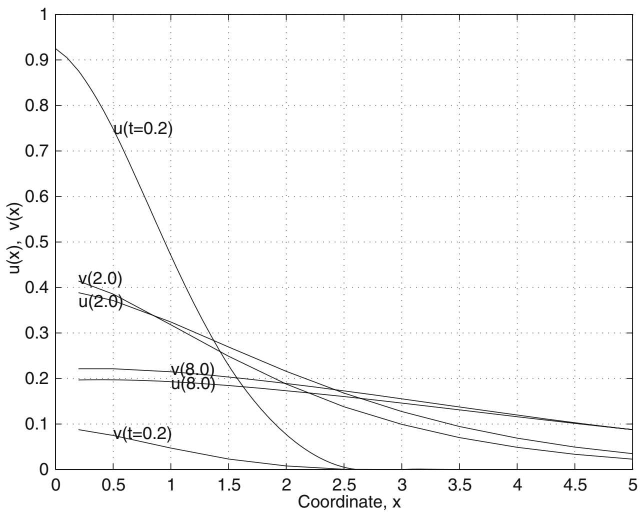

Fig. 2 Curves $u(x, t)$ and $v(x, t)$ labeled with values of dimensionless time t. Example with $\epsilon = 0.2$ and $\beta = 1$

and

$$Q_1(x, t) = \frac{2\beta^{3/2}}{\sqrt{1-\epsilon}} \int_0^\infty \exp\left(-\frac{\beta}{1-\epsilon}\frac{\zeta^2}{\beta z^2+\zeta^2}\right) \times \cos\left(\zeta\sqrt{\frac{\beta\epsilon}{1-\epsilon}\frac{z^2}{\beta z^2+\zeta^2}+1}\right)\frac{d\zeta}{(\beta z^2+\zeta^2)^{3/2}}, \tag{28}$$

where $z = x/D_*$.

For the calculation of $v(x, t)$, it is convenient to employ Eq. 13.

The results of calculations of an example with $\epsilon = 0.2$ and $\beta = 1$ are presented in Fig. 2.

For the limiting case, describing an instantaneous source in homogeneous medium, $\epsilon = 1$, Eqs. 26 and 27 reduce to Eq. 24. In this solution, the total NAPL quantity in the mound

$$\Omega(t) = \int_0^\infty u(x, t)dx \tag{29}$$

remains a constant independent of time. For the general case, $\epsilon \leq 1$, with the boundary condition $u(0, t) = 1/\sqrt{\pi t}$, the integrals

$$\Omega_u = \int_0^\infty u(x, t)dx \quad \text{and } \Omega_v = \int_0^\infty v(x, t)dx \tag{30}$$

are dependent on time, as it is seen from an example presented in Fig. 3 for the same parameter values as in Fig. 2. The value Ω_u changes from some non-zero value at $t = 0$ to other (larger) finite value at $t \to \infty$. The value Ω_v varies from zero at $t = 0$ to some finite value at $t \to \infty$.

Discussing the behavior of Ω_u and Ω_v we must consider the time dependence of the value of $\partial u(0, t)/\partial x$, which is proportional to the inflow of NAPL into the mound at $x = 0$. For the limiting case of a mound in homogeneous porous medium ($\epsilon = 1$), initiated by instantaneous source, $\partial u(0, t)/\partial x$ is a δ-function of time, and $\Omega_u(t)$ is a constant. In the general case, as it is seen from the example in Fig. 3, $\Omega_u(0) = \Omega^o > 0$, and at $\to \infty$, $\Omega_u(t) \to \Omega_1 > \Omega_o$.

The function $\partial u(0, t)/\partial x$ may be obtained using Laplace transform. It consists of δ-function of time with intensity $\sqrt{\epsilon}$ at $t = 0$ and an additional function:

$$\psi(t) = \frac{\beta}{2\sqrt{\epsilon}} \exp\left(-\frac{\beta(1+\epsilon)}{2\epsilon(1-\epsilon)}t\right)\left[I_0\left(\frac{\beta}{2\epsilon}t\right)+I_1\left(\frac{\beta}{2\varepsilon}t\right)\right] \tag{31}$$

where $I_0(\tau)$ and $I_1(\tau)$ are modified Bessel functions of the first kind. The function $\psi(t)$ is a decreasing one, $\psi(0) = \beta/2\sqrt{\epsilon}$, and $\psi(t) \to 0$ at $t \to \infty$.

In the limiting case $\epsilon = 1$, the function $\psi(t)$ is identical to zero, and the inflow is described by $\delta-$function. In the general case, at $t = 0$, the instantaneous inflow proportional to $\sqrt{\epsilon}$ takes place, and at $t > 0$, there is additional positive inflow described by a decreasing function $\psi(t)$. The entire inflow, described by the function $\psi(t)$, namely,

$$\int_0^\infty \psi(t)dt,$$

is equal to $1 - \sqrt{\epsilon}$. Consequently, the total inflow into the mound is equal to 1, the same as in the case of homogeneous soil.

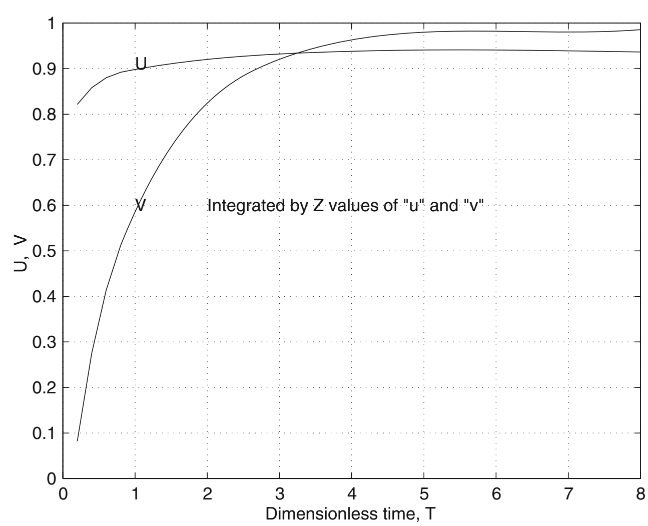

Fig. 3 Curves $u(x, t)$ and $v(x, t)$ integrated in $z = x/D_o$ from 0 to ∞, as a function of t

3.2 Axisymmetric flow

In the case of plane radial flow, Eqs. 10 and 11 transform to

$$\epsilon\frac{\partial u}{\partial t}-\frac{D_{\mathrm{o}}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\beta(u-v)=0,$$
$$(1-\epsilon)\frac{\partial v}{\partial t}-\beta(u-v). \tag{32}$$

Applying the Laplace transform to Eq. 32, we obtain for the image U the equation

$$\frac{D_{\mathrm{o}}}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right)=\left(\epsilon+\frac{(1-\epsilon)\beta}{(1-\epsilon)p+\beta}\right)pU \tag{33}$$

and, for V, expression (13).

The solution of Eq. 33, vanishing at $t\to\infty$ is

$$U(r,p)=C_1K_0\left(\frac{r}{\sqrt{D_0}}\sqrt{p\lambda(p)}\right),$$
$$\lambda(p)\equiv\epsilon+\frac{(1-\epsilon)\beta}{(1-\epsilon)p+\beta}. \tag{34}$$

$K_0(z)$ is a zero-order modified Bessel function of the imaginary argument. In the limiting case of homogeneous media, $\epsilon=1$, $\lambda=1$, the function $K_0(r\sqrt{p/D_{\mathrm{o}}})$ is the Laplace transform of the known fundamental solution of the linear diffusion equation:

$$u_0(r,t)=\frac{1}{2t}\exp\left(-\frac{r^2}{D_{\mathrm{o}}t}\right), \tag{35}$$

corresponding to the instantaneous source with unit intensity. For this solution, the initial distribution $u(r,0)$ may be expressed as $\delta-$ function of $r/\sqrt{D_{\mathrm{o}}}$.

The solution of system (32) may be obtained by inversion of function (34). Same as in the linear case, we transform integral (16) to an integral along the negative part of the abscissa in the $p-$plane.

Final expression for $u(r,t)$ obtained using also the relationships for Bessel functions looks as follows:

$$u(r,t)=S_{r1}+S_{r2}, \tag{36}$$

where

$$S_{r1}=\int_0^{\sqrt{\beta/(1-\epsilon)}}\exp(-t\xi^2)$$
$$\times J_0\left(\xi\sqrt{\epsilon+\frac{\beta(1-\epsilon)}{\beta-\xi^2(1-\epsilon)}}\frac{r}{\sqrt{D_0}}\right)\xi\mathrm{d}\xi \tag{37}$$

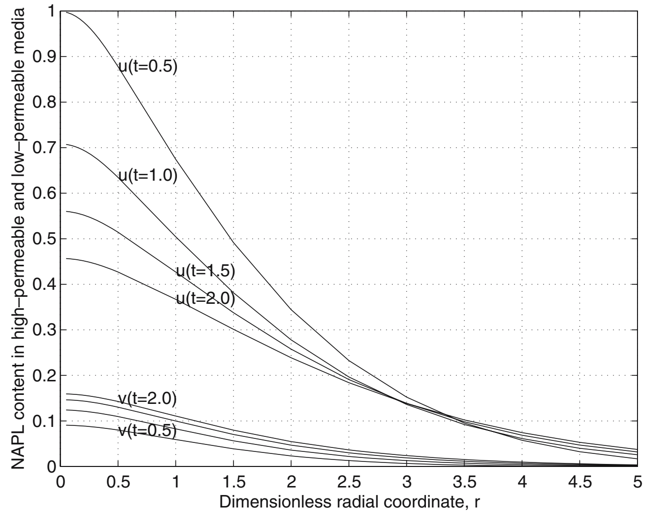

Fig. 4 Distribution of values $u(r,t)$ and $v(r,t)$ around an instantaneous source

and

$$S_{r2}=\int_{\sqrt{\beta/\epsilon/(1-\epsilon)}}^{\infty}\exp(-t\xi^2)$$
$$\times J_0\left(\xi\sqrt{\epsilon-\frac{\beta(1-\epsilon)}{\xi^2(1-\epsilon)-\beta}}\frac{r}{\sqrt{D_0}}\right)\xi\mathrm{d}\xi. \tag{38}$$

$J_0(x)$ is a Bessel function of zero order.

In the limiting case of homogeneous medium ($\epsilon=1$), the integral S_{r2} vanishes and the integral S_{r1} reduces to

$$\int_0^{\infty}\exp(-t\xi^2)J_0\left(\frac{r}{\sqrt{D_o}}\xi\right)\xi\mathrm{d}\xi,$$

which can be found in the tables of integrals by Gradstein and Ryzhik [3] and is equal to expression (35).

For calculation of $v(r,t)$, Eq. 13 can be used with the change of x to r.

The results of example calculations of $u(r,t)$ and $v(r,t)$ are presented in Fig 4. The values $\beta=1$ and $\epsilon=0.2$ were used in the calculations.

In Fig. 5, the distribution of total NAPL content in a mound, $q(r,t)=\epsilon u(r,t)+(1-\epsilon)v(r,t)$, is shown as a function of time.

As it is seen from examples in Figs. 4 and 5,

$$\frac{\partial u(0,t)}{\partial r}=0\quad\text{and}\quad\frac{\partial v(0,t)}{\partial r}=0. \tag{39}$$

This property can be established by direct differentiation of Eqs. 37 and 38. This means that at $t>0$, there is no flow of NAPL into the mound at the origin; the

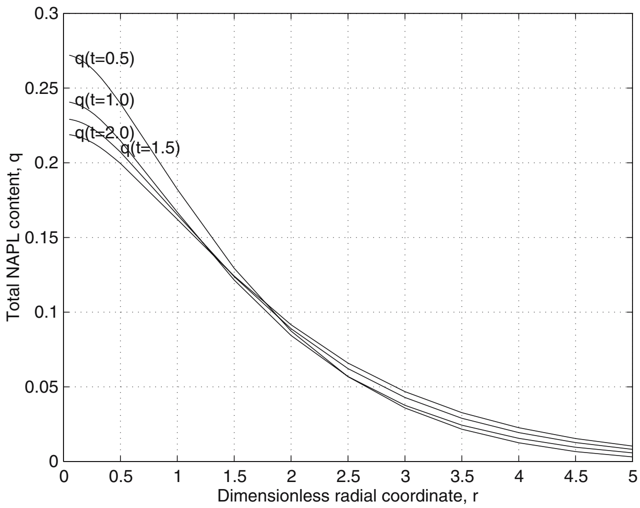

Fig. 5 Distribution of total NAPL content $q(r, t)$ for the same example as in Fig. 4

source is instantaneous. The total NAPL content in the mound

$$Q(t) = \epsilon \int u(r, t) r dr + (1 - \epsilon) \int v(r, t) r \mathrm{d}r$$

remains constant.

3.3 Flow of a mound above a sloping plane

If a DNAPL mound is moving along a plane surface with a relatively large value of i_o, i.e.

$$i_o \gg \left| \frac{\partial u}{\partial x} \right|, \tag{40}$$

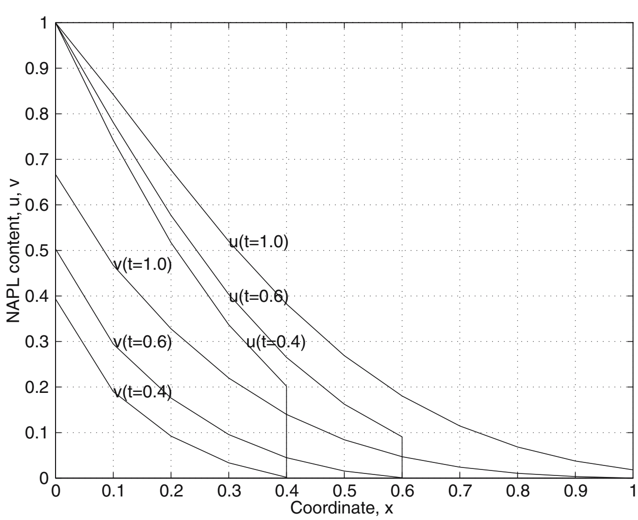

Fig. 6 Distribution of $u(\eta, T)$ and $v(\eta, T)$ in a lens moving along a sloping plane surface. The values of T are shown near the curves

one can neglect the second derivative in Eq. 9 and obtain a linear equation

$$\epsilon \frac{\partial u}{\partial t} + C \frac{\partial u}{\partial x} + \beta (u - v) = 0, \tag{41}$$

where $C = D_1 i_o$.

We consider the following problem for the hyperbolic system of Eqs. 10 and 41, namely, NAPL lens formation in infinite one-dimensional aquifer from a constant linear source:

$$\text{at } t = 0,\ u = 0, \quad v = 0, \quad \text{at} \quad x = 0,\ u = 1. \tag{42}$$

The system of Eqs. 10 and 41 for $u(x, t)$ and $v(x, t)$ has two families of characteristics: $t = t_0 = \text{const}$ and $x - Wt = \text{const}$, where $W = C/\epsilon$. The front of NAPL is spreading from the source at $x = 0$ with the velocity $W = C/\epsilon$. There is no NAPL at all points $x > Wt$.

The solution of the problem Eq. 42 can be found using Laplace transform.

The solution for $u(x,t)$ may be obtained using the tables by Bateman and Erdelyi in the form (for $x < Wt$):

$$u(\eta, T) = \exp\left(-\frac{1-\epsilon}{\epsilon}\eta\right)\left[\exp(-T+\eta)) I_0\left(2\sqrt{\frac{1-\epsilon}{\epsilon}\eta(T-\eta)}\right)\right.$$
$$\left. + \int_0^{T-\eta} \exp(-\tau) I_0\left(2\sqrt{\frac{1-\epsilon}{\epsilon}\eta\tau}\right) \mathrm{d}\tau\right] \tag{43}$$

where

$$\eta = \frac{\beta \epsilon u}{C(1-\epsilon)}, \quad T = \frac{\beta}{1-\epsilon} t.$$

At $\eta > T$, i.e. at $x > Wt$, $u(x, T) \equiv 0$.

The function $v(x, t)$ using Eqs. 13 and 16.

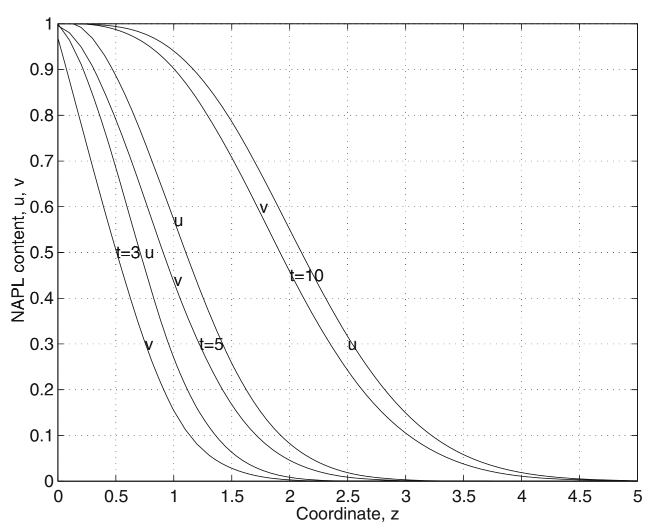

Fig. 7 Same as in the previous figure for greater values of T

At relatively small values of T, expression (43) gives a non-zero value of u at $\eta = T$, which means that a jump in u exists at the "front" of NAPL. The distributions of $u(\eta, T)$ and $v(\eta, T)$ were calculated for some examples.

In Figs. 6 and 7, the distributions of u and v are shown for the case $\epsilon = 0.2$. The jumps of $u(\eta, T)$ are seen at the curves for $T < 1$ at $\eta = T$. If the value of ϵ is close to 1, the jumps exist at much greater values of T than in the case of small ϵ.

4 Conclusions

The water or NAPL mound spreading in a double-porosity medium above an impermeable surface has been modeled using a linearized mathematical model. The equations for NAPL content in HP and LP components (fractures and matrix) were derived as a generalization of the equations describing a water mound or NAPL lens movement above impermeable plane surface in a homogeneous porous medium with the assumption of vertical gravity–capillary equilibrium. The heterogeneous soil is assumed to be described by double-porosity model of soil consisting of alternating HP and LP domains with mass transfer from HP to LP zones. The equations for flow in double-porosity soil were linearized, and the Laplace transform was used to obtain analytical solutions of a number of example problems. The solutions describe the mound spreading in terms of NAPL content in both domains as functions of time and spatial coordinates. They include one-dimensional and axisymmetric problems of lens formation from a source and spreading of a mound originated from an instantaneous source over a horizontal surface. For the flow above a sloping surface, the one-dimensional NAPL or water front propagation in the HP and LP media formed a line with a constant initial NAPL content in high-permeable soil. In most of the examples, the fraction of HP domains is chosen to be much less than the fraction of LP ones. Such model has properties close to porous-fractured soil. In all the examples, the delay of average NAPL content is observed as compared with the mound spreading in homogeneous soil. At the same time, the spreading of NAPL in the HP medium is much faster than in the LP one. The delay in the NAPL distribution as compared with the spreading in a homogeneous medium with average permeability increases with the permeability of HP component.

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