Journal of Hydrology 243 (2001) 205–215
Journal of Hydrology
www.elsevier.com/locate/jhydrol

A simple analytical expression to describe tidal damping or amplification

H.H.G. Savenije[a,b]

[a]IHE-Delft, P.O. Box 3015, 2601 DA Delft, The Netherlands
[b]TU-Delft, P.O. Box 5048, 2600 GA Delft, The Netherlands

Received 16 June 2000; revised 30 October 2000; accepted 1 December 2000

Abstract

Can a single line describe tidal damping or amplification? Tidal damping and amplification in alluvial estuaries are constrained by an implicit feedback mechanism. The time lag ϵ between the occurrence of high water and high water slack is crucial in this regard. An analytical solution of the St. Venant equations appears to result in a surprisingly simple explicit relation for the tidal range, consisting of an exponential and a linear term. In alluvial estuaries, the linear term is dominant, particularly in the case of tidal amplification. In the case of tidal damping the exponential term only becomes important in the upper reaches of the estuary (preventing the expression for the tidal range from becoming negative). In tidal amplification, the exponential term is suppressed by the newly defined tidal Froude Number which (as it contains sin ϵ) tends to zero when the tidal wave gets a predominantly standing wave character. This negative feedback prevents the development of an exponentially increasing tidal range. Finally, the expression obtained is a very useful explicit equation to determine estuary parameters that are difficult to determine from direct observations, such as the roughness and the mean water depth.

Keywords: Tidal damping; Alluvial estuaries; Tidal hydraulics; Analytical solution; Tidal Froude number

1. Introduction

The most intriguing thing that comes up as one carries out surveys in estuaries, is that the water world that we observe appears much more regular and much simpler than our non-linear differential equations suggest. Moreover, close scrutiny of the physical laws reveals chaotic processes and potentially unstable behaviour which we seldom observe at the relevant scales of water engineering. Certainly there is a lot of chaos in nature. At the scale of the water particle there is turbulence, salt dispersion, chaotic groundwater flow and unpredictable behaviour of rainfall–runoff processes. But as soon as we zoom out, these very processes show a regular and predictable pattern of motion which can be described by relatively simple mathematical "laws", such as Manning's friction term in St. Venant's equation for open channel flow, the dispersion term in the advection–dispersion equation, Darcy's equation for groundwater flow, and linear reservoirs in rainfall–runoff processes.

When, some 20 years ago, the author started to survey estuaries, it appeared surprising that in alluvial estuaries there was no apparent bottom slope; that the longitudinal width varied according to an exponential function; that along the estuary the maximum tidal velocity at spring tide was never more than 1 m/s,

E-mail address: hsa@ihe.nl (H.H.G. Savenije).

0022-1694/01/$ - see front matter
PII: S0022-1694(00)00414-5

and also not much less; that the tidal excursion (the distance that a water particle travels during a tidal cycle) was constant along the estuary and in the order of 10 km at spring tide for a semi-diurnal tide (and double that amount for a diurnal tide); that the tidal volume (or tidal prism) equals the product of the tidal excursion and the tidal-average cross-sectional area; and that tidal damping or amplification in estuaries was modest or non-existent. Although these "facts" are common knowledge to an experienced surveyor of estuaries, they by no means appear to be a direct result from the non-linear conservation of mass and momentum equations that we use in hydraulics. Hence, in summary, we know our equations. We know our hydraulic theory. We know that this theory is not simple or trivial. Yet we observe a relatively simple and predictable world. How can that be explained?

Now it appears that this is no exception. At larger temporal and spatial scales, chaotic processes tend to become predictable. A good example is Boyle's gas law. Chaotic behaviour of individual gas molecules culminates into a straightforward relation between pressure, volume and temperature. The same applies to Ohm's law, where chaotic behaviour of electrons boils down to a simple relation between current, friction and potential, completely in line with Darcy's law for groundwater flow. And there are many more examples. In fact, virtually all physical processes become chaotic when we look at them in finer detail. Several researchers have tried to enhance their understanding of physical processes by going into more detail, by looking at smaller time scales and smaller spatial scales, only to become disappointed.

From this perspective, the observation that tidal damping in estuaries is very modest and predominantly linear is an interesting topic for further investigation. This behaviour is not at all obvious since the equations on which damping is based are non-linear and can very well develop into exponential behaviour. In fact, as will be demonstrated, longitudinal damping appears to be a combination of a linear term and an exponential term, wherein the linear term predominates. This makes the phenomenon rather unspectacular. Most researchers (Ippen, 1966; Savenije, 1992a,b, 1993), however, assumed exponentially decreasing or increasing tidal ranges. So, where does the predominantly linear term result from? In the next section the derivations are made. In Section 3 the theory is applied to a set of real estuaries. Section 4 presents a sensitivity analysis from which we can draw the interesting conclusion that the sensitivity to certain parameters is quite different in the estuaries studied.

2. Derivation of the new equation for tidal damping

The general equation for tidal damping was derived by Savenije (1998) from the complete St. Venant equations using a Lagrangean reference frame. The basis for the derivation is that the longitudinal width variation along the estuary axis is exponential and that the average tidal depth is constant along the estuary. These assumptions with regard to estuary shape correspond to observations in alluvial estuaries (provided the alluvial plane is sufficiently large) which were justified by Savenije (1992a, pp. 53–58). Several authors make use of these assumptions in their derivations, among whom are Hunt (1964), Ippen (1966), McDowell and O'Connor (1977), and Savenije (1986). Furthermore, for the derivations it is required that the ratio of tidal range to depth (H/h) is less than unity. In Appendix A, a brief summary of the derivation is presented. Analytical solution of the St. Venant equations yields:

$$\frac{h}{H}\frac{\mathrm{d}H}{\mathrm{d}x}\left(1+\frac{gH}{2cv\sin\epsilon}\right)=\frac{h}{b}-f\frac{g}{C^2}\frac{v\sin\epsilon}{c} \qquad (1)$$

where, h is the tidal-average estuary depth (m), H the tidal range (m), g the acceleration due to gravity (9.81 m/s^2), c the observed celerity of the tidal wave (m/s), v the amplitude of the tidal velocity (m/s), ϵ the phase lag in radians between high water (HW) and high water slack (HWS), b the length scale of the exponential width variation (m), C Chezy's roughness (m$^{0.5}$/s) and f a friction factor accounting for the difference in average water depth during ebb and flood flow (defined in Appendix A).

The assumptions made with regard to estuary shape correspond to an ideal estuary as described by Pillsbury (1956). In an ideal estuary, there is no tidal damping. The increase of energy per unit width of the incoming tidal wave as the banks converge is exactly compensated by friction loss. Consequently, width varies exponentially, and depth, roughness and wave celerity are constant along the estuary. From Eq.

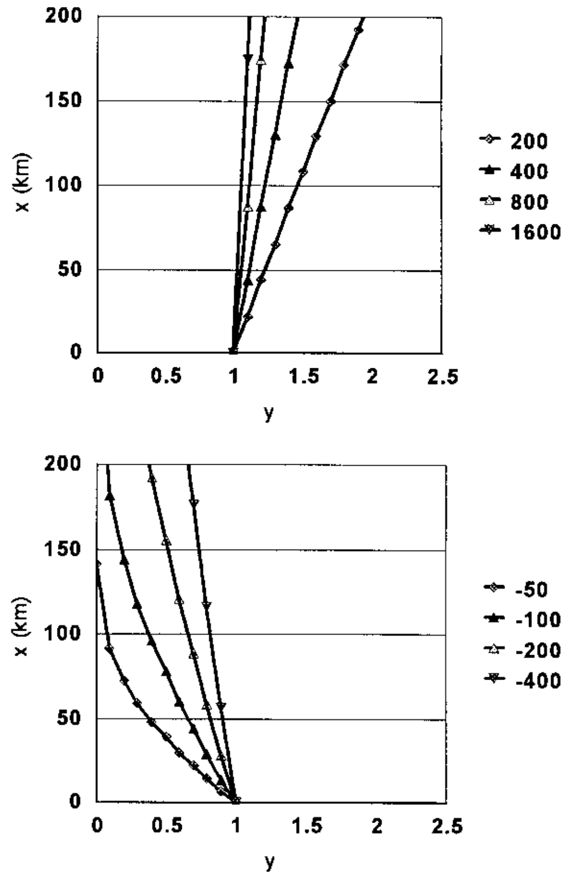


Fig. 1. (a) Tidal amplification as a function of different (positive) values of β/α. (b) Tidal damping as a function of different (negative) values of β/α.

(1), it can be seen that in an ideal estuary:

$$\frac{h}{b} = f\frac{g}{C^2}\frac{\upsilon \sin \epsilon}{c} \tag{1a}$$

where all parameters are constant along the estuary. In practice, this condition does not always hold true. Most estuaries experience some form of damping or amplification, albeit modest. What appears to hold, however, is the assumption that the width varies exponentially and that there is a modest or zero bottom slope. The exceptions to this "rule" are estuaries where there is non-alluvial forcing (rock outcrops) or where the alluvial plain has not had sufficient opportunity to develop (often caused by the absence of a coastal shelf).

To simplify Eq. (1), three parameters are introduced, the dimensionless tidal range y, the dimensionless tidal Froude number α, and the tidal damping scale β which has a length dimension:

$$y = \frac{H}{H_0} \tag{2}$$

$$\alpha = \frac{2c\upsilon \sin \epsilon}{gH_0} \tag{3}$$

$$\frac{1}{\beta} = \frac{1}{b} - \frac{fg}{C^2}\frac{\upsilon \sin \epsilon}{ch} \tag{4}$$

where H_0 is the tidal range at $x = 0$. The tidal Froude number α differs from the regular Froude number in that it contains the tidal range to depth ratio (H/h) and the Wave-type number ($\sin \epsilon$), introduced by Savenije (1998). This number equals zero in case of a purely standing wave and equals unity in case of a purely progressive wave. In alluvial estuaries, the tidal wave is of a mixed character where $0 < \sin \epsilon < 1$. The tidal damping scale β is the length scale of exponential tidal damping, which is negative in the case of tidal damping (when friction outweighs amplification due to the conversion of the banks), and positive in the case of tidal amplification. The two parameters α and β contain parameters which according to the studies performed earlier by Savenije (1992a,b, 1993, 1998) may be considered constant along the longitudinal axis of the estuary.

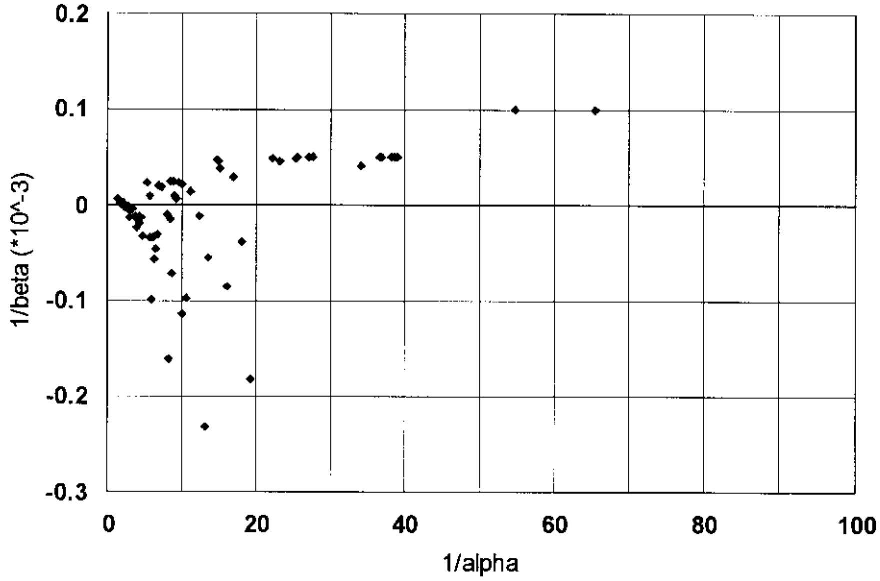
Substitution of y, α and β into Eq. (1) leads to the following differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(1 + \frac{1}{\alpha}y\right) = \frac{1}{\beta}y \tag{5}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha}{\beta}\frac{y}{(\alpha + y)} \tag{5a}$$

It can be see from Eq. (5a) that when $\alpha \gg y$, the tidal range increases or decreases exponentially with x at a length scale equal to β. However, if $\alpha \ll y$, the increase or decrease is linear with a factor α/β. The order of magnitude of y is unity, which implies that if $\alpha \ll 1$, $\mathrm{d}y/\mathrm{d}x$ is constant (and damping or amplification is linear). When, as a result of tidal damping y approaches zero, even if



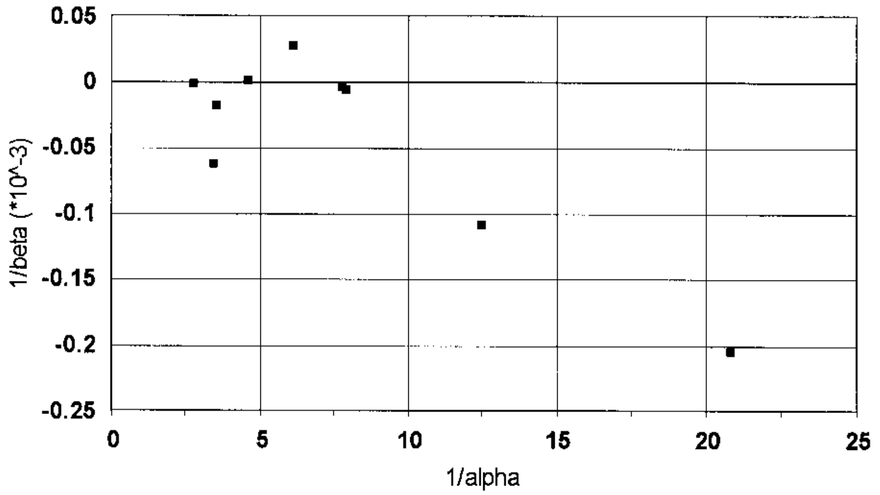


Fig. 2. (a) Values of $1/\alpha$ and $1/\beta$ in simulated estuaries. (b) Values of $1/\alpha$ and $1/\beta$ in real estuaries.

$\alpha \ll 1$, then damping becomes exponential, preventing y from intersecting the x-axis. It can also be seen that if β approaches infinity, the tidal range is constant. Hence whether damping or amplification is large or small depends on the value of α/β: if α/β is close to 0, damping is modest or linear; if α/β is large, negatively or positively, damping or amplification is exponential.

Eq. (5a) can be integrated by separation of variables (personal communication by E.J.M. Veling, 1999):

$$dx = \left(\frac{\beta}{y} + \frac{\beta}{\alpha}y\right) dy \tag{5b}$$

With $y = 1$ ($H = H_0$) at $x = 0$, integration yields:

$$x = \beta \ln(y) + \frac{\beta}{\alpha}(y - 1) \tag{6}$$

This is a very simple relation consisting of a logarithmic

Table 1
Characteristics of estuaries analysed

	H_0 (m)	b (km)	h (m)	v (m/s)	c (m/s)	C ($m^{0.5}$/s)	$\sin \epsilon$	$f \times g$ (m/s^2)	α	β (km)	β/α (km)
Schelde	4.9	28	10	1.1	10	70	0.5	10.6	0.22	42	187
Gambia	1.2	56	8.7	0.9	6	60	0.4	10.0	0.37	−550	−1494
Pungue	6	20	4.3	0.9	4	40	0.4	19.5	0.05	−5	−102
Lalang	3	217	10.6	0.9	7	60	0.3	10.2	0.13	−175	−1391
Chao Phya	1.8	109	8	0.9	7	50	0.4	10.1	0.28	−59	−212
Tha Chin	2.7	87	5.3	0.9	3	45	0.4	10.7	0.08	−9	−116
Limpopo	1.1	50	7	0.8	5	60	0.3	10.1	0.22	1199	5495
Incomati	1	42	3	0.9	4	60	0.4	10.3	0.29	−16	−56
Maputo	2.8	16	3.6	0.9	5	60	0.4	11.8	0.13	3000	25,000

and a linear term. The only complication in Eq. (6) is that the dependent variable y cannot be written as an explicit function of x. Fig. 1a and b presents plots of this relation (for positive and negative values of β/α), where y and x are plotted on the horizontal and vertical axes, respectively. The case where $\alpha/\beta = 0$ (β/α goes to infinity), is the special case of the ideal estuary, where there is no damping ($y = 1$).

Although Eq. (6) cannot be written as $y = f(x)$, one can see that if β/α is large, the first term becomes less important and Eq. (6) is closely reproduced by the simple line:

$$y = \frac{H}{H_0} = 1 + \frac{\alpha}{\beta}x \tag{7}$$

It appears that in natural estuaries β/α is large. In fact, β/α is the length scale of the linear damping (or amplification) process, whereas β is the length scale of the exponential process. Since $\alpha < 1$ in all alluvial estuaries, tidal amplification and damping are predominantly linear. Fig. 2 shows a plot of the values of $1/\beta$ and $1/\alpha$ that were obtained in a wide range of simulated estuaries (see Savenije 1992a,b, 1993, 1998). In all cases simulated: $\beta/\alpha > 200$ km for amplification and $\beta/\alpha < -50$ km for tidal damping. With these values, the error made by using Eq. (7) instead of Eq. (6) is small. Eq. (7) represents a bundle of lines through the point (0, 1), where the tidal range $H = H_0$ at $x = 0$. The exponential term, $\beta \ln(y)$, only becomes important in tidal damping when y approaches zero, and consequently α is no longer small as compared to y. When the line of Eq. (7) approaches the x-axis, the exponential term, which gains importance, prevents it from intersecting the x-axis. Hence Eq. (7) loses its applicability near the point where $x = -\beta/\alpha$. As long as that point lies far enough upstream from the area under consideration, Eq. (7) is applicable.

2.1. The importance of the phase lag between HWS and HW

In the relative importance of the two terms of Eq. (6), the phase lag ϵ between HWS and HW plays a key role. In estuary hydraulics, tidal waves are always of a mixed type where $0 < \epsilon < \pi/2$ (Savenije, 1992a, pp. 23–25). A purely progressive wave, wherein $\epsilon = \pi/2$, only occurs in prismatic channels (where b tends to infinity) and where β is negative and dominated by the friction term. Furthermore it results in a relatively large value of α. The result is exponential damping, which one normally observes in rivers and canals where progressive waves occur. A purely standing wave occurs when $\epsilon = 0$. As a result, $\alpha = 0$, $\beta = b$, and tidal amplification is linear but is approaching zero. In estuaries, it appears that ϵ prevents the development of exponential amplification through an implicit feedback mechanism. In estuaries where strong tidal amplification occurs, the value of ϵ becomes small, decreasing the value of α, so that, in turn, the exponential amplification is suppressed. Fig. 2 shows that for a β-value of 10 km ($1/\beta = 0.1 \times 10^{-3}$), which otherwise would imply strong amplification, α assumes a value of less than 2% ($1/\alpha > 50$). The ratio β/α then equals 500 km, which is a very large length scale for amplification. Hence, the reduction of α, through a reduction of ϵ, largely undoes the amplification.

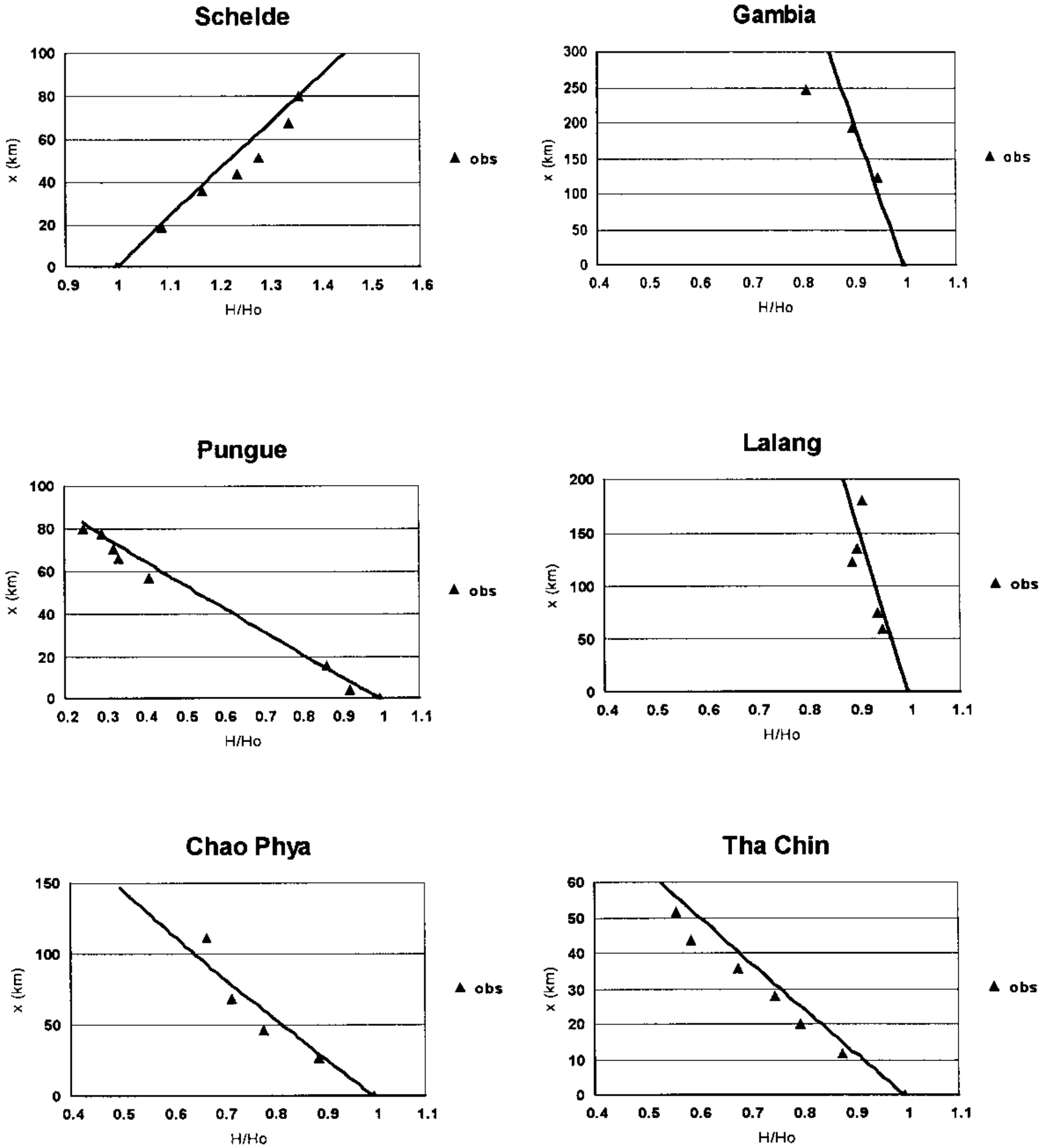


Fig. 3. (a–i) Observed and computed longitudinal variation of tidal range in selected estuaries.

3. Application of the derived formula to observations

The next step is to confront the newly found formula with observations in real estuaries. In the past, the author carried out many surveys in estuaries, mostly in Africa and Asia, but these observations had a different purpose than to study tidal damping; in fact they were done to observe salt intrusion. As a result, not always were the relevant parameters observed. In

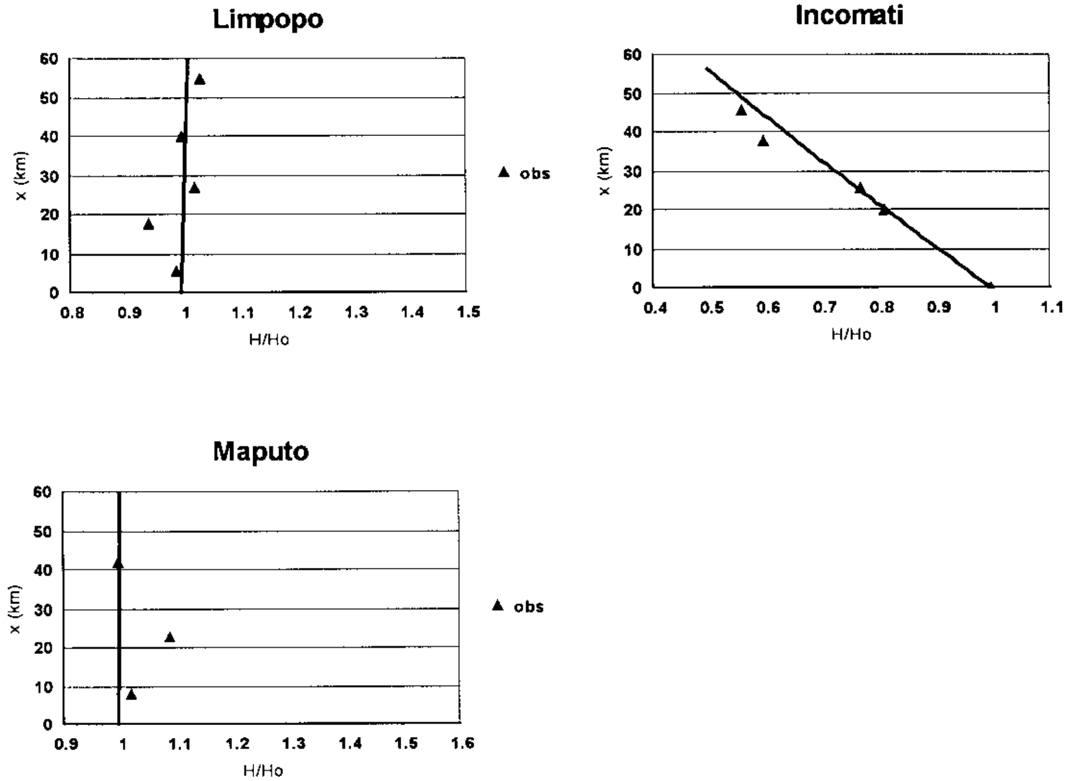


Fig. 3. (*continued*)

Table 1, a selection is presented of the nine estuaries of which tidal damping observations were available, and that contained most of the relevant information. In this table the values of H_0, b, h, and c were observed accurately. The values of ϵ, υ, and C had to be estimated (ϵ and υ) or calibrated (C) by fitting Eq. (6) to the observed longitudinal tidal damping, as is presented in Fig. 3a–i.

The tidal velocity amplitude was nowhere observed accurately, but earlier work has shown that at spring tide the value is in the order of 1 m/s in alluvial estuaries (Bretting, 1958; Bruun and Gerritsen, 1960; Savenije, 1992a, p. 156). Where the observed tidal range was less than spring tide, the value of the velocity amplitude was reduced accordingly. For $\sin \epsilon$ a value was assumed close to 0.4, which corresponds to about 45 min time lag during a semi-diurnal tide, which has been observed in most estuaries. In the Schelde the lag time is close to 1 h, which is rather long for a semi-diurnal tide. Also, the velocity amplitude in the Schelde is larger than expected, which may be the result of dredging. These values were nowhere observed systematically or accurately, and given the crucial role that the phase lag plays in containing tidal amplification, this is a field of research that merits much more field observation in the future. The roughness is a notoriously difficult parameter to observe directly, and is generally found by calibration. In alluvial estuaries, where bottom roughness is generally low (fine material and modest bed forms), relatively high values of C (around 60) are often found.

Although the values obtained in Table 1 may be subject to refinement through observations, this does not affect the longitudinal fitting of the equations in Fig. 3a–i. It can be seen from these figures that the

Table 2
Sensitivity of y to a relative error in H, b, h, c, C^2, and $\upsilon \sin \epsilon$, respectively, at a distance $x = 50$ km from the estuary mouth, for different estuaries

Estuary	$\alpha x/\beta$	dH/H	db/b	dh/h	dc/c	dC^2/C^2	$d(\upsilon \sin \epsilon)/\upsilon \sin \epsilon$	dy
Schelde	0.22	−0.23	−0.29	0.07	0.29	0.07	0.15	0.06
Gambia	−0.03	0.02	−0.32	0.35	0.32	0.35	−0.37	0.34
Pungue	−0.49	−0.69	−0.12	0.61	0.12	0.61	−1.10	−0.57
Lalang	−0.04	0.03	−0.03	0.06	0.03	0.06	−0.10	0.06
Chao Phya	−0.24	0.23	−0.13	0.36	0.13	0.36	−0.60	0.36
Tha Chin	−0.43	0.36	−0.05	0.48	0.05	0.48	−0.91	0.41
Limpopo	0.01	−0.01	−0.22	0.21	0.22	0.21	−0.20	0.21
Incomati	−0.89	0.82	−0.34	1.23	0.34	1.23	−2.13	1.16
Maputo	−0.02	−0.13	−0.40	0.42	0.40	0.42	−0.44	0.27

theory fits the observed tidal damping very well. Results are satisfactory in all classes of estuary: (1) an estuary where substantial tidal amplification occurs (Schelde); (2) estuaries with substantial tidal damping (Pungue, Tha Chin, Incomati); (3) estuaries with modest tidal damping (Gambia, Lalang, Chao Phya); and (4) estuaries with no tidal damping (Limpopo, Maputo).

4. Sensitivity analysis

Given the uncertainty of the calibrated values in the previous section, it appears worthwhile to carry out a sensitivity analysis on the unknown parameters of Table 1. Although Eq. (6) is analytical and straightforward, the sensitivity to the individual parameters is not directly clear. The complicating factor is the subtraction in Eq. (4), which causes the relative importance of the different variables to vary strongly between estuaries. In some estuaries, for example, the sensitivity to $\upsilon \sin \epsilon$ is large, whereas in others it is insignificant.

For the sensitivity analysis, the following equation applies:

$$dy = c\frac{\partial y}{\partial c}\frac{dc}{c} + b\frac{\partial y}{\partial b}\frac{db}{b} + h\frac{\partial y}{\partial h}\frac{dh}{h} + C^2\frac{\partial y}{\partial C^2}\frac{dC^2}{C^2} + H\frac{\partial y}{\partial H}\frac{dH}{H} + \cdots \quad (8)$$

Using the linear approximation of Eq. (7), further elaboration of Eq. (8) yields:

$$dy = \frac{\alpha x}{\beta}\left[\frac{\beta}{b}\left(\frac{dc}{c} - \frac{db}{b}\right) + \left(\frac{\beta}{b} - 1\right) \times \left(\frac{dh}{h}\left(1 + \frac{f}{2}\left(\frac{H}{h}\right)^2\right) + \frac{dC^2}{C^2} - \frac{f}{2}\left(\frac{H}{h}\right)^2\frac{dH}{H}\right) - \frac{dH}{H} + \left(2 - \frac{\beta}{b}\right)\frac{d(\upsilon \sin \epsilon)}{\upsilon \sin \epsilon}\right] \quad (9)$$

It is important to note that during amplification ($\beta > 0$) the terms in ($\beta/b - 1$) and ($2 - \beta/b$) counteract each other and hence reduce the sensitivity to h, C^2, and $\upsilon \sin \epsilon$, whereas during tidal damping ($\beta < 0$) the sensitivity to the uncertainty in these parameters is enhanced. The latter is particularly apparent in the case of the Pungue, Tha Chin and Incomati, which experience considerable damping.

Table 2 provides an overview of the sensitivities to errors in the input parameters at a relevant distance of $x = 50$ km. From Table 2 it can be concluded that in the estuaries that experience strong damping (Pungue, Incomati and Tha Chin) the overall sensitivity to errors is large, as α/β is large. In general, the sensitivity to relative errors in $\upsilon \sin \epsilon$ is largest, whereas this parameter is not monitored systematically. This is an important recommendation for further research. Of the remaining parameters, H_0, b, and c are easy to determine with accuracy. The parameter which is hard to determine is h, to which the expression also has a high sensitivity. Finally, C generally follows from calibration since it is never measured directly. However, we know that in estuaries, its value generally ranges between 40 and 60 $m^{0.5}/s$.

5. Conclusions

In alluvial estuaries there is no exponential amplification of the tidal range along the estuary axis. If there is tidal amplification, then the amplification is almost completely linear. There appears to be a negative feedback in tidal propagation that prevents the occurrence of exponential amplification of the tidal range. In this feedback, the phase lag ϵ between HW and HWS is instrumental. Tidal amplification is dominant in estuaries where there is strong convergence, and where the tidal wave develops a "standing wave" character, so that the tidal-wave number, $\sin\epsilon$, approaches zero. It can be seen from Eq. (4), that this would lead to maximum amplification (the shortest amplification length scale: $\beta = b$). However, if the tidal-wave number approaches zero, then α, the tidal Froude number, approaches zero as well. This, in turn, makes the tidal amplification predominantly linear, according to Eq. (6), and even reduces tidal amplification to zero, according to Eq. (5a).

In estuaries where there is tidal damping, the process is also predominantly linear, particularly in the reach close to the estuary mouth, where y is close to unity. As one moves further inland, the tidal range decreases and gradually the exponential term becomes dominant as y approaches zero. In estuaries with tidal damping, the process is primarily exponential upstream from the point where $x = -\beta/\alpha$; downstream from that point the damping is primarily linear.

The fact that the tidal-wave number is a crucial parameter in tidal propagation, makes it a prime variable to be observed during hydrometric surveys in estuaries. Yet the phase lag between HW and HWS is seldom observed systematically. This requires an adjustment of measuring protocols.

Finally, fitting Eq. (6) to observations of tidal propagation is an indirect way of measuring roughness and average estuary depth. Knowing that in estuaries C ranges between 40 and 70, this method can provide an estimate of average estuary depth, which is otherwise difficult to obtain.

Acknowledgements

The author wishes to thank Dr E.J.M. Veling of TU-Delft for his valuable suggestions in the derivation of Eq. (6). The very valuable comments of the reviewers are highly appreciated.

Appendix A. Derivation of the tidal damping equation

The following derivation is a summary and elaboration of the derivations in Savenije (1998). The derivation is based on an exponentially varying width:

$$B = B_0 \exp\left(-\frac{x}{b}\right) \tag{A.1}$$

where B is the average cross-sectional and tidal-average width, B_0 is the width at the estuary mouth, x is the distance from the mouth, and b is the convergence length, the length scale of the width convergence. The derivation is further based on the complete St. Venant equations for one-dimensional flow, with the positive x-axis pointing in an upstream direction. In an Eulerian reference system, they can be written as:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} + gI_b + g\frac{v|v|}{C^2 h} = 0 \tag{A.2}$$

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{A.3}$$

where $v(x,t)$ is the cross-sectional average flow velocity, $h(x,t)$ the cross-sectional average depth of flow, g the acceleration due to gravity, I_b the bottom slope, C Chezy's coefficient for resistance to flow, $Q(x,t)$ the tidal discharge, and $A(x,t)$ the cross-sectional area.

In a Lagrangean approach, the dependent variables of Eqs. (A.2) and (A.3) are determined in a moving reference frame that moves with the velocity of flow:

$$v = \frac{dx}{dt} \tag{A.4}$$

According to Savenije (1992b), the conservation of mass equation (A.3) when written in a Lagrangean form, and after substitution of Eq. (A.1) and the assumption that $B\ \partial h/\partial t \gg h\ \partial B/\partial t$, can be modified into:

$$\frac{\mathrm{d}h}{h} = \left(\frac{1}{b} - \frac{1}{\gamma}\right)\mathrm{d}x + \frac{1}{c}\,\mathrm{d}v \tag{A.5}$$

with

$$\frac{1}{\gamma} = \frac{1}{\upsilon}\frac{\mathrm{d}\upsilon}{\mathrm{d}x} \approx \frac{1}{H}\frac{\mathrm{d}H}{\mathrm{d}x} \tag{A.6}$$

where γ is the length scale of the damping of the tidal velocity amplitude $\upsilon(x)$, which according to Savenije (1993), by approximation, can be assumed to be equal to the length scale of the damping of the tidal range $H(x)$.

The combination of Eqs. (A.4)–(A.6), and further elaboration yields:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{c}{h}\frac{\mathrm{d}h}{\mathrm{d}t} - \frac{cv}{b} + \frac{cv}{H}\frac{\mathrm{d}H}{\mathrm{d}x} \tag{A.7}$$

The momentum equation (A.2), when written in a Lagrangean reference frame, yields:

$$\frac{\mathrm{d}v}{\mathrm{d}t} + g\frac{\partial h}{\partial x} + gI_{\mathrm{b}} + g\frac{v|v|}{C^2 h} = 0 \tag{A.8}$$

The combination of Eqs. (A.7) and (A.8), and using Eq. (A.4) yields:

$$\frac{cv}{gh}\frac{\mathrm{d}h}{\mathrm{d}x} - \frac{cv}{g}\left(\frac{1}{b} - \frac{1}{H}\frac{\mathrm{d}H}{\mathrm{d}x}\right) + \frac{\partial h}{\partial x} + I_{\mathrm{b}} + \frac{v|v|}{C^2 h} = 0 \tag{A.9}$$

The tidal range H is defined as:

$$H = h_{\mathrm{HW}} - h_{\mathrm{LW}} \tag{A.10}$$

where h_{HW} is the water depth at high water (HW) and h_{LW} is the depth at low water (LW). Subsequently:

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\mathrm{d}h_{\mathrm{HW}}}{\mathrm{d}x} - \frac{\mathrm{d}h_{\mathrm{LW}}}{\mathrm{d}x} \tag{A.11}$$

Both for HW and LW, by definition:

$$\frac{\partial h}{\partial t} = 0 \tag{A.12}$$

and hence:

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\partial h}{\partial x} \tag{A.13}$$

If the tidal wave is un-deformed (which is the case when $H/h \ll 1$), the damping is symmetrical with respect to the average water level, which has a slope I:

$$\frac{\mathrm{d}h_{\mathrm{HW}}}{\mathrm{d}x} + \frac{\mathrm{d}h_{\mathrm{LW}}}{\mathrm{d}x} \approx I \tag{A.14}$$

$$h_{\mathrm{HW}} \approx h + H/2 \tag{A.15}$$

$$h_{\mathrm{LW}} \approx h - H/2 \tag{A.16}$$

For the tidal velocity at HW and LW, Savenije (1993) derived the following expressions:

$$v_{\mathrm{HW}} = \upsilon\sin(\epsilon) \tag{A.17}$$

$$v_{\mathrm{LW}} = -\upsilon\sin(\epsilon) \tag{A.18}$$

where ϵ is the phase difference between the occurrence of HWS and HW (or LWS and LW).

The combination of Eqs. (A.9), (A.13), (A.15) and (A.17) yields for HW:

$$\frac{c\upsilon\sin\epsilon}{g(h+H/2)}\frac{\mathrm{d}h_{\mathrm{HW}}}{\mathrm{d}x} - \frac{c\upsilon\sin\epsilon}{g}\left(\frac{1}{b} - \frac{1}{H}\frac{\mathrm{d}H}{\mathrm{d}x}\right) + \frac{\mathrm{d}h_{\mathrm{HW}}}{\mathrm{d}x} + \frac{(\upsilon\sin\epsilon)^2}{C^2(h+H/2)} = -I_{\mathrm{b}} \tag{A.19}$$

Similarly for LW:

$$\frac{-c\upsilon\sin\epsilon}{g(h-H/2)}\frac{\mathrm{d}h_{\mathrm{LW}}}{\mathrm{d}x} + \frac{c\upsilon\sin\epsilon}{g}\left(\frac{1}{b} - \frac{1}{H}\frac{\mathrm{d}H}{\mathrm{d}x}\right) + \frac{\mathrm{d}h_{\mathrm{LW}}}{\mathrm{d}x} - \frac{(\upsilon\sin\epsilon)^2}{C^2(h-H/2)} = -I_{\mathrm{b}} \tag{A.20}$$

Subtraction of Eqs. (A.19) and (A.20) yields:

$$\frac{c\upsilon\sin\epsilon}{g(h+H/2)}\left(\frac{\mathrm{d}h_{\mathrm{HW}}}{\mathrm{d}x} + \frac{\mathrm{d}h_{\mathrm{LW}}}{\mathrm{d}x}\frac{(h+H/2)}{(h-H/2)}\right) - \frac{2c\upsilon\sin\epsilon}{gh}\left(\frac{h}{b} - \frac{h}{H}\frac{\mathrm{d}H}{\mathrm{d}x}\right) + \frac{\mathrm{d}H}{\mathrm{d}x} + \frac{2f}{h}\frac{(\upsilon\sin\epsilon)^2}{C^2} = 0 \tag{A.21}$$

with

$$f = \left(\frac{1}{1 - \left(\frac{H}{2h} \right)^2} \right) \tag{A.22}$$

where f is a friction factor ($f > 1$), which enhances the effect of friction to cater for the fact that the friction term is larger at LW than at HW, particularly if H/h is approaching unity.

The part between brackets in the first term of Eq. (A.21) can be replaced by I of Eq. (A.14), provided $H/h \ll 1$. Elaboration yields:

$$\frac{h}{H}\frac{\mathrm{d}H}{\mathrm{d}x}\left(1 + \frac{gH}{2c\upsilon \sin \epsilon}\right) = \frac{h}{b} - f\frac{g}{C^2}\frac{\upsilon \sin \epsilon}{c} - \frac{I}{(2 + H/h)} \tag{A.23}$$

Since during periods of low river discharge I is at least an order of magnitude 10 smaller than h/b, the last term can be disregarded in most practical cases, leading to Eq. (1) of the main text. Further research into the relative importance of I versus h/b is needed, which is complicated by the fact that I is difficult to observe accurately.

Appendix B. Nomenclature

Symbol definition (dimensions: L, length; T, time)

A	Cross-sectional area [L^2]
b	Convergence length [L]
B	Cross-sectional mean stream width [L]
B_0	Width at the estuary mouth [L]
c	Wave celerity [L/T]
C	Chezy's coefficient [$L^{0.5}$/T]
F	Tidal Froude number (υ/c)
f	Friction factor accounting for the difference in friction at LW and HW
g	Acceleration due to gravity [L/T^2]
h	Stream depth [L]
h_{HW}	Stream depth at HW [L]
H	Tidal range [L]
H_0	Tidal range at the estuary mouth [L]
I	Tidal-average water level slope
I_b	Bottom slope
Q	Tidal discharge [L^3/T]
t	time [T]
v	Tidal velocity [L/T]
v_{HW}	Tidal velocity at HW [L/T]
y	Dimensionless tidal range
z	Dimensionless tidal range
x	Distance [L]
α	Tidal Froude number
β	Tidal damping scale [L]
γ	Length scale of tidal velocity damping [L]
ϵ	Phase difference between HWS and HW;
υ	Tidal velocity amplitude [L/T]

References

Bretting, A.E., 1958. Stable channels. Acta Polytecnica Scandinavia 245. Copenhagen, Denmark.

Bruun, P., Gerritsen, F., 1960. Stability of Coastal Inlets. North-Holland, Amsterdam.

Hunt, J.N., 1964. Tidal oscillations in estuaries. Geophysical Journal of the Royal Astronomical Society 8, 440–455.

Ippen, A.T., 1966. Estuary and Coastline Hydrodynamics. McGraw-Hill, New York.

McDowell, D.M., O'Connor, B.A., 1977. Hydraulic Behaviour of Estuaries. The Macmillan Press, London.

Pillsbury, G., 1956. Tidal hydraulics. Corps of Engineers, Vicksburg, USA.

Savenije, H.H.G., 1986. A one-dimensional model for salinity intrusion in alluvial estuaries. Journal of Hydrology 85, 87–109 (Elsevier, Amsterdam, The Netherlands).

Savenije, H.H.G., 1992. Rapid assessment technique for salt intrusion in alluvial estuaries. PhD thesis, IHE Report Series, No. 27. International Institute for Infrastructure, Hydraulics and Environment, Delft, The Netherlands.

Savenije, H.H.G., 1992b. Lagrangian solution of St Venant's equations for an alluvial estuary. Journal of Hydraulic Engineering, ASCE 118 (8), 1153–1163.

Savenije, H.H.G., 1993. Determination of estuary parameters on the basis of Lagrangian analysis. Journal of Hydraulic Engineering, ASCE 119 (5), 628–643.

Savenije, H.H.G., 1998. Analytical expression for tidal damping in alluvial estuaries. Journal of Hydraulic Engineering 124 (6), 615–618.